Symmetry and Completeness in Relevant Epistemic Logic

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Abstract

In this paper, we provide an axiom system for the relevant logic of equivalence relation frames and prove completeness for it. This provides a partial answer to the longstanding open problem of axiomatizing frames for relevant modal logics where the modal accessibility relation is symmetric. Following this, we show that the logic enjoys Halldén completeness and that a related logic enjoys the disjunction property.

1 Introduction

Recently the project of combining ideas and techniques from relevant logics with those from epistemic logic has attracted a lot of attention.[1] Many of the developments in this project diverge from the common modelling technique of using equivalence relations to interpret the knowledge operator, as in many classically-based epistemic logics.[2,3] In contrast to these, Standefer (2025a) studied the logic of frames that use

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[1] For more on relevant logics, the reader should see the introductions and overviews by Dunn and Restall (2002), Read (1988), Mares (2020), Bimbó (2007), Logan (2024), and Standefer (2024; 202x). Rendsvig et al. (2023) provide a reader-friendly overview of epistemic logics.

[2] See Meyer and van der Hoek (1995), Stalnaker (2006), or van Ditmarsch et al. (2007), for examples of the use equivalence relations on modal frames, what we will call 'partition frames' below.

[3] There have been many developments in relevant epistemic logics, broadly understood, including those by Bílková et al. (2010), Sedlár (2015; 2016), Bílková et al. (2016), Savíc and Studer (2019), Standefer (2019; 2023a; 2023c), Tedder and Bílková (2022), Punčochář et al. (2023), Ferenz (2024a), Sedlár and Vigiani (2023; 2024), and Vigiani (2024).

a modal accessibility relation S that is an equivalence relation. Perhaps the most salient philosophical virtue of these frames is that the use of the equivalence relations allows one to employ the same epistemic indistinguishability interpretation on points used in partition frames, namely Kripke frames whose modal accessibility relation S is an equivalence relation, for classically-based epistemic logics.[4] According to this interpretation, the members of an equivalence class are those worlds that are epistemically indistinguishable to the agent given their background information. The logic of partition frames is $\mathsf{S5}$, and there are many criticisms directed as $\mathsf{S5}$ as an epistemic logic. Standefer argued that Eq, the logic of equivalence frames for a relevant logic, avoids many of those criticisms.[5] The logic Eq contains neither the so-called negative introspection axiom, $(\mathsf{5})$ $\neg\Box A \to \Box\neg\Box A$, nor the (B) axiom, $A \to \Box\neg\Box\neg A$, which axioms are some of the main targets of criticisms in the context of epistemic logics.[6] In addition, Eq avoids the (K) axiom, $\Box(A \to B) \to (\Box A \to \Box B)$, and the (Nec) rule, from A being a logical truth to infer $\Box A$ is, both of which are forms of logical omniscience.[7] Thus, the logic of equivalence frames is promising as an epistemic logic, both because (i) it avoids some of the philosophical criticisms directed at the use of classical partition frames while maintaining some of their virtues and (ii) it provides an appealingly simple contrast to other relevant epistemic logics and their frames.

No axiomatization of Eq was given by Standefer (2025a). Axiomatizing the logic of equivalence frames presents some challenges. Since the modal accessibility relation S is an equivalence relation, it is symmetric, i.e. if Sab then Sba, which might suggest the use of (B) in its axiomatization. Whereas the (B) axiom is valid on classical Kripke frames where S is symmetric, the (B) axiom is invalid on equivalence frames for relevant logics. There are relevant modal logics that contain the (B) axiom and are complete with respect to an appropriate class of frames, as shown by Fuhrmann (1990). Those frames, however, do not obey the symmetry condition for S, in general, unlike the case with classically-based modal logics containing (B). The condition

[4] When discussing classical Kripke frames, it is more common to use 'R' rather than 'S' for the modal accessibility relation. In this paper, 'R' is used for the ternary relation in the frames for relevant logics, so we are using 'S' for the different modal accessibility relations.

[5] Standefer (2025a) focused on the logic of equivalence relations over frames for the relevant logic R, below we focus on the logic of equivalence relations over frames for the weaker logic B. Nothing hinges on the choice of base logic at least with respect to completeness and the epistemic interpretation discussed above. Some of the further results we prove do depend on choice of base logic.

[6] Hintikka (1962) and Williamson (2000) are both prominent sources for those criticisms. The criticisms focused on the positive introspection axiom, $(\mathsf{4})$, are not avoided.

[7] The logic is closed under the rule (Mono), from $A \to B$ being a logical truth to infer $\Box A \to \Box B$ is, so the logic does not avoid all forms of omniscience.

associated with (B) is

- if Sab, then Sb^*a^*,

where * is the Routley star, used to interpret negation. Since, in general, x^* need not be identical to x, this condition will not deliver symmetry for S.[8] Mares (1994) shows that on frames with the additional condition, that if Sab, then Sa^*b^*, the addition of the (B) axiom to a logic secures completeness with respect to frames where S is symmetric, as it does in classically-based modal logics. Mares's result, however, does not settle the issue with respect to equivalence frames. First, this is a substantive condition that is not adopted by many logicians working in the area; and, second, in the context of the equivalence frames of Standefer (2025a), imposing that condition yields a proper subclass of frames, what Standefer calls "coordinated equivalence frames," and it results in both (B) and (5) being valid, which undermines the distinctiveness of the equivalence frames in the setting of frames for relevant logics.[9] Given the philosophical differences between the logic of equivalence frames and the logic of coordinated equivalence frames, having an axiomatization of the former is important.

Further, having an axiomatization for frames where with a symmetric modal accessibility relation S is desirable for at least one additional reason: Symmetry is a natural condition that plays an important role in many common structures, such as the equivalence relations used in partition frames for classically-based epistemic logics. In this paper, we will provide a sound and complete axiomatization of the logic of equivalence frames, filling the gap left by Standefer (2025a). Our axiomatization contains the axiom, $\Box(A \vee \Box B) \to (\Box A \vee \Box B)$, which is valid in S5, but it does not use (B).[10] We prove completeness with respect to equivalence frames using a canonical model construction for a logic containing this axiom. Our proof uses additional axioms to secure the symmetry condition on the modal accessibility relation of the canonical frame, so there are questions remaining about axiomatizing frames where S is symmetric. In addition to the completeness result, we use other techniques to show that this logic and a close relative have other good features, so this paper is, in part, a technical companion to the philosophical discussion of Standefer (2025a).

[8] Cf. the discussion of this point by Standefer (2023b).

[9] Coordinated equivalence frames appear briefly below, in definition 40.

[10] It does not use (5) either, which in the present setting is interderivable with (B), given (T). We will return to (B) and (5) below.

It is worth observing that with classical logic as the base logic, (5) is interderivable with the axiom we call (M5) below, $\Box(A \vee \Box B) \to (\Box A \vee \Box B)$, given (K) and (Nec). This is exercise 4.37(a) from Chellas (1980). We would like to thank Rohan French for this pointer.

The plan for this paper is as follows. In §2, we will present the required background on the frames and axiom systems for non-modal relevant logics. In §3, we present the relevant modal logic BEq that will be our focus along with equivalence frames. Then, we prove completeness for that logic with respect to the equivalence frames of the previous section (§4). We then, in §5, use metavaluations to prove some metatheoretic results about a slight extension of BEq, in particular that the extension has the disjunction property. After that, in §6, we show that BEq is Halldén complete. Following this, we explore the features of the defined possibility operator in the models (§7). Finally, in §8, we close with a short summary and an open question.

2 Background

We will work in a propositional language $\mathcal{L}$ with a countably infinite set of atoms, $\mathsf{At} = \{p, q, r, \ldots\}$. The logical connectives are the vocabulary $\{\neg, \rightarrow, \wedge, \vee, \Box\}$, and in later sections of the paper we will briefly consider other singulary modal operators, $\mathcal{B}$ and $\mathcal{K}$. The $\Box$ operator will not come into the picture until the next section.

We will use ternary relational frames, augmented by modal accessibility relations. We begin with the basic frames for the relevant logic B.

Definition 1 (Frame)**.** *A* B*-frame is a tuple* $\langle K, N, R, {}^*\rangle$ *obeying the following conditions, where for* $a, b \in K$*,* $a \leq b$ *is defined as* $\exists x \in N : Rxab$*.*

(i) If $a \leq b$*, then* $b^* \leq a^*$*.*

(ii) $a^{**} = a$*.*

(iii) If $a \in N$ *and* $a \leq b$*, then* $b \in N$*.*

(iv) If $d \leq a$*,* $e \leq b$*, and* $c \leq f$ *and* $Rabc$*, then* $Rdef$*.*

It is worth noting that condition (iv) is slightly stronger than what is needed below, although this condition does enable easy extension with connectives such as fusion.[11] As usual, one can obtain frames for stronger logics by imposing additional conditions on the frames. We will not go further into those details here. Instead, we will proceed to define models and validity.

Definition 2 (Model)**.** *A* B*-model* M *is a* B*-frame* $\langle K, N, R, {}^*\rangle$ *together with a valuation function* $V : \mathsf{At} \mapsto \wp K$ *such that if* $a \in V(p)$ *and* $a \leq b$*, then* $b \in V(p)$*.*

Such a model is said to be built on *the frame* $\langle K, N, R, {}^*\rangle$*.*

[11] We thank an anonymous referee for pointing this out.

We inductively define truth at a world for the whole language.

- $a \Vdash p$ iff $a \in V(p)$
- $a \Vdash \neg B$ iff $a^* \nVdash B$
- $a \Vdash B \wedge C$ iff $a \Vdash B$ and $a \Vdash C$
- $a \Vdash B \vee C$ iff $a \Vdash B$ or $a \Vdash C$
- $a \Vdash B \to C$ iff for all $b, c \in K$, if $Rabc$ and $b \Vdash B$, then $c \Vdash C$

Given this definition, we can define notation for propositions.

Definition 3 (Proposition)**.** *In a given model M, $[\![A]\!] = \{a \in K : a \Vdash A\}$.*

This is useful notation that will facilitate some of our later discussion.

Definition 4 (Holding, validity)**.** *A formula A* holds in a model *M iff for all $b \in N$, $b \Vdash A$.*

A formula A is valid on a frame *F iff for all models M built on F, A holds in M.*

A formula A is valid in a class $\mathcal{C}$ of frames *iff for all frames $F \in \mathcal{C}$, A is valid on F.*

We write $\models_{\mathcal{C}} A$ when A is valid in $\mathcal{C}$.

We write $\models_{\mathsf{B}} A$ when A is valid in the class of all B*-frames.*

Next, we note, without proof, two standard lemmas that we will appeal to in our proofs. The appeals will be left implicit, as they are standard.

Lemma 5 (Heredity lemma)**.** *For all formulas A, if $a \Vdash A$ and $a \leq b$, then $b \Vdash A$.*

Lemma 6 (Verification lemma)**.** *The following are equivalent in a given model M, for all formulas B and C.*

- *For all $b \in K$, if $b \Vdash B$, then $b \Vdash C$.*
- *For all $a \in N$, $a \Vdash B \to C$.*

The axiom system for B is the following.[12] The final five principles are rules, with the formulas to the left of '$\Rightarrow$' being the premises of the rule and the formula to the right the conclusion.

[12] Throughout we will view logics in the framework FMLA, that is as sets of formulas. See Humberstone (2011, 103ff.) for more on logical frameworks.

(Ax1) $A \to A$

(Ax2) $(A \wedge B) \to A$, $(A \wedge B) \to B$

(Ax3) $A \to (A \vee B)$, $A \to (B \vee A)$

(Ax4) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

(Ax5) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

(Ax6) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

(Ax7) $\neg\neg A \to A$

(Ru1) $A, A \to B \Rightarrow B$

(Ru2) $A, B \Rightarrow A \wedge B$

(Ru3) $A \to \neg B \Rightarrow B \to \neg A$

(Ru4) $A \to B \Rightarrow (C \to A) \to (C \to B)$

(Ru5) $A \to B \Rightarrow (B \to C) \to (A \to C)$

B is the smallest set of formulas containing all the axioms and closed under the rules.

Definition 7 (Proof)**.** *A* B-proof *is a sequence of formulas each of which is either an axiom or is the conclusion of a rule whose premises occur earlier in the sequence.*

The final member of a B*-proof is the* conclusion *of the proof.*

We write $\vdash_{\mathsf{B}} A$ *when* A *is the conclusion of a* B*-proof.*

The axiom system for B is sound and complete with respect to the frames for B.

Theorem 8. $\vdash_{\mathsf{B}} A$ *iff* $\models_{\mathsf{B}} A$*.*

Proof. For a proof, see Routley et al. (1982, ch. 4). □

Since we are only discussing logics in this paper that are sound and complete with respect to an indicated class of frames, we will use the term 'theorem' indifferently between members of a set defined in terms of proofs and a set defined in terms of frames.

Before proceeding to the modal logics, we will note that we are focusing on the logic B, which is a natural, weak relevant logic. It is the logic of the ternary relational frames with the fewest conditions.[13] As mentioned above, one can obtain stronger logics by imposing conditions on the class of frames and by adding axioms (or rules) to the axiomatization. The completeness result of our paper will go through even with strengthening the base logic. Some of the other results, in particular the metacompleteness result, depend on specific features of the base logic, so we will focus on B. Below we will mention the logic R, which is perhaps the best known relevant logic. It was central to the work of Anderson and Belnap (1975) and was defended at length by Mares (2004). Perhaps most salient to our purposes is that it was the focus of the discussion of Standefer (2025a). To obtain R from B, one can add the following axioms.[14]

(R1) $(A \to \neg B) \to (B \to \neg A)$

(R2) $A \to ((A \to B) \to B)$

(R3) $(A \to B) \to ((C \to A) \to (C \to B))$

(R4) $(A \to (A \to B)) \to (A \to B)$

Corresponding to each of these axioms, there are frame conditions.

(F1) If $Rabc$, then $Rac^{*}b^{*}$

(F2) If $Rabc$, then $Rbac$

(F3) If $\exists x \in K(Rabx$ and $Rxcd)$, then $\exists x \in K(Rbxd$ and $Racx)$

(F4) If $Rabc$, then $\exists x \in K(Rabx$ and $Rxbc)$

Soundness and completeness extends to these axioms with respect to frames satisfying these conditions. We will not need to rely on any features of R below, but these details are included to make the discussion more self-contained.

[13] There are weaker logics one can get on these frames, such as BM, whose frames drop the involution postulate on the star. BM was studied by Fuhrmann (1990), as well as more recently by Ferguson and Logan (2024). Weaker logics, such as BB, require alternative frames, such as those discussed by Goble (2003) or Ferenz and Tedder (2023).

[14] Brady (1984) provides more details on axiomatizations for relevant logics, including R, as well as Fitch-style natural deduction systems.

3 Modal logics

To obtain an equivalence frame for B, we equip the B-frames with an equivalence relation S on K.

Definition 9 (Equivalence frame)**.** *An equivalence frame for* B *is a tuple* $\langle K, N, R, S, ^* \rangle$ *such that* $\langle K, N, R, ^* \rangle$ *makes up a* B*-frame and the following conditions are satisfied.*

(i) If $a \leq b$ *and* Sbc*, then there is* $x \in K$ *such that both* $x \leq c$ *and* Sax*.*

(ii) S *is an equivalence relation on* K*.*

The definitions of model, counterexample are extended to equivalence frames in the straightforward way. For a given frame, we will define the notation $[a] = \{b \in K : Sab\}$.

Now we can give the distinctive principles for the modal operator, $\Box$. Following each principle, we list a common name for it, if any.

(M1) $A \to B \Rightarrow \Box A \to \Box B$ (Mono)

(M2) $(\Box A \wedge \Box B) \to \Box(A \wedge B)$ $(\wedge\Box)$[15]

(M3) $\Box A \to A$ (T)

(M4) $\Box A \to \Box\Box A$ (4)

(M5) $\Box(A \vee \Box B) \to (\Box A \vee \Box B)$

The logic resulting from adding these principles to B we will call BEq. The first two principles, a rule and axiom, are found in all relevant modal logics obtained from classes of frames with a binary modal accessibility relation. The next two principles are the relatively well-known principles (T) and (4). The final one is less well-known and deserves further comment. Axiom (M5) is, like all the axioms above, valid on classical Kripke frames, called 'partition frames' below, for S5, and it plays an important role in the sequent system for S5 due to Ohnishi and Matsumoto (1957).[16] Ono (1977) discusses it in the context of intuitionistic S5-type logics, but it has otherwise received little attention.

In the present setting, axioms (M5) arguably captures what is logically distinctive about the equivalence frames. Indeed, adapting the work of Standefer (2023b), we

(1) $A \to B \Rightarrow \Box A \to \Box B$ (Mono)	(5) $\Box(A \to B) \to (\Box A \to \Box B)$ (K)
(2) $(\Box A \wedge \Box B) \to \Box(A \wedge B)$ ($\wedge\Box$)	(6) $A \to \Box\neg\Box\neg A$ (B)
(3) $\Box A \to A$ (T)	(7) $\neg\Box\neg A \to \Box\neg\Box\neg A$ (5)
(4) $\Box A \to \Box\Box A$ (4)	(8) $A \Rightarrow \Box A$ (Nec)

Table 1: BS5 modal principles

could define the logic BS5 as B extended by the principles in table 1.[17] BS5 is a good point of comparison with BEq. This is because BS5 is incomparable with BEq while also being a plausible S5-type extension of B.[18]

Before proceeding, it will be useful to have the classical Kripke frames for S5 on the table.

Definition 10 (Partition frame, universal frame)**.** *A* partition frame *is a pair $\langle W, S\rangle$ where $W \neq \emptyset$ and S is an equivalence relation on W.*

A universal frame *is a partition frame where $S = W \times W$.*

The definitions of model and validity and the truth conditions for the classical connectives are all standard, so we omit them here. Since we are going to be concerned only with the frames where S is an equivalence relation, we adopt the name 'partition frame', to distinguish them from the more general Kripke frames that permit alternative accessibility relations. As is well known, both the class of partition frames and the narrower class of universal frames generate the logic S5, which can be axiomatized in classical logic using the modal principles for BS5 from table 1, understanding the implication classically, and an axiomatization of classical logic.[19] Standefer (2023b) showed that these three presentations of S5 diverge when translated over to relevant logics.

[15]This is also called (C) by Chellas (1980) and ($\Box$C) by Fuhrmann (1990)

[16]In the present context of Routley-Meyer frames with a modal accessibility relation, (M5) is valid on frames where S is Euclidean, meaning that if Sab and Sac, then Sbc, whereas (5) is not.

[17]We note that this axiomatization is somewhat redundant. The redundancies are to facilitate comparisons.

[18]These are not the only plausible candidates for S5-type extensions of B. Standefer (2023b; 2025a) discusses others, and Ferenz (2024b) identifies another, based on a correspondence with monadic quantified logic.

[19]See Mints (1992), Humberstone (2016), or Garson (2018) for details and discussion.

While BS5 is, in many ways, a good candidate for an S5-type logic over B, it fails to have many common S5 theorems among its theorems, such as (M5), so it is worth considering other candidate logics.

Lemma 11. *The axiom* (M5) *is not a theorem of* BS5*.*

Proof. A countermodel can be obtained using John Slaney's program MaGIC.[20] We leave finding a countermodel to the interested reader.

□

One can strengthen the base logic from B to R, if one wants, and it will not result in (M5) becoming derivable. Thus, the extension of B with the principles (Mono), ($\wedge\Box$), (T), and (4) is the common core of BEq and BS5. One can strengthen that logic in two diverging ways, with one adding (M5) and the other adding (B), (5), and (Nec).

There are frames for BS5, and associated soundness and completeness results. We do not need to present details of those frames here.[21] One consequence of the preceding theorem, which we want to emphasize, is that the frames for BS5 are not, in general equivalence frames. Some of them can supply counterexampless to (M5), which rules out being equivalence frames. While BS5 does contain the (B) axiom, the corresponding frame condition for that is: if Sab, then Sb^*a^*. This condition, in general, falls short of symmetry. This is a point to which we return in the final section, and it is crucial for seeing the novelty in the present work.

We will close this section by proving the soundness of BEq with respect to equivalence frames for B. While most of the principles were shown sound by Standefer (2025a), albeit in the context of equivalence frames for the logic R, we will reprove the soundness of (M5) here.

Lemma 12. *Let M be a model on an equivalence frame for* B*. If Sab, then $a \Vdash \Box A$ iff $b \Vdash \Box A$.*

Proof. Suppose Sab. Since S is an equivalence relation, $[a] = [b]$. Suppose $a \Vdash \Box A$. This is the case iff $[a] \subseteq [\![A]\!]$, which is equivalent to $[b] \subseteq [\![A]\!]$. But this is $b \Vdash \Box A$, as desired. The converse is similar. □

Theorem 13. *For all $a \in K$, if $a \Vdash \Box(A \vee \Box B)$ then $a \Vdash \Box A \vee \Box B$.*

[20] See Slaney (1995). Logan (2024, 15ff.) provides a tutorial on the use of MaGIC.

[21] The interested reader can consult the discussion of RS5 by Standefer (2023b; 2025a) or the general discussion of modal frames by Fuhrmann (1990).

Proof. Let a be an arbitrary world in K. Suppose that $a \Vdash \Box(A \vee \Box B)$ and that $a \nVdash \Box A$. Then there is some world b such that Sab and $b \nVdash A$. We show that $a \Vdash \Box B$. Suppose not. Lemma 12 shows that the same $\Box$-formulas are true at a and b. Thus, $b \nVdash \Box B$. But then $a \nVdash \Box(A \vee \Box B)$. So, by reductio, $a \Vdash \Box B$ as required. $\Box$

It will be useful to prove a lemma demonstrating that an implication we will need is, in fact, a theorem of BEq. To justify some of the steps of that lemma, as well as some of the moves we will make in the next section, we will list, without proof, some rules that hold for many relevant logics, including BEq.

Lemma 14. *The logic* BEq *is closed under the following rules.*

- $A \to B, B \to C \Rightarrow A \to C$
- $A \to (B \vee C), C \to D \Rightarrow A \to (B \vee D)$

Lemma 15. $(\Box A \vee \Box B) \to \Box(\Box A \vee \Box B)$ *is a theorem of* BEq.

Proof. The derivation is as follows.

(1) $\Box A \to (\Box A \vee \Box B)$, by (Ax3)

(2) $\Box\Box A \to \Box(\Box A \vee \Box B)$, by (1) and (Mono)

(3) $\Box A \to \Box\Box A$, by $(\mathsf{4})$

(4) $\Box A \to \Box(\Box A \vee \Box B)$, by (2), (3), and lemma 14

(5) $\Box B \to \Box(\Box A \vee \Box B)$, similar to the preceding

(6) $(\Box A \vee \Box B) \to \Box(\Box A \vee \Box B)$, by (Ru1), (Ru2), and (Ax5)

$\Box$

We will need this formula for a step in one of the parts of the completeness proof of the next section.

4 Completeness

In this section we will prove the completeness of BEq with respect to equivalence frames for B. We will do this by using a variation on the Henkin-style canonical model construction commonly used to show completeness for relevant logics. Some of the steps and lemmas are well-known, so we will not reprove them here.[22] We will, instead, focus on the changes that need to be made in order to carry the proof through, as well as definitions needed to make the proof self-contained.

Let us begin by defining some important concepts and stating a standard lemma. Then, we will define the canonical frame.

Definition 16 (Theory, prime theory). *A set* Γ *of formulas is a* BEq-theory *iff both (i) whenever* $A \to B$ *is a theorem of* BEq *and* $A \in \Gamma$*, then* $B \in \Gamma$*, and (ii) if* $A \in \Gamma$ *and* $B \in \Gamma$*, then* $A \wedge B \in \Gamma$*.*

A theory Γ *is* prime *iff whenever* $A \vee B \in \Gamma$*, then either* $A \in \Gamma$ *or* $B \in \Gamma$*.*

Definition 17 (Inconsistent pair). *Let* Σ *and* Δ *be theories.* $\Sigma \vdash \Delta$ *iff there are* $A_1, \ldots, A_n \in \Sigma$ *and* $B_1, \ldots, B_m \in \Delta$ *such that* $(A_1 \wedge \ldots \wedge A_n) \to (B_1 \vee \ldots \vee B_m)$ *is a theorem of* BEq*.*

If $\Sigma \vdash \Delta$*, we say that* (Σ, Δ) *is a* BEq*-inconsistent pair.*

A pair of set of formulas (Σ, Δ) *is a* BEq*-consistent pair iff it is not a* BEq*-inconsistent pair.*

Lemma 18 (Belnap-Gabbay Extension Lemma). *If* (Σ, Δ) *is a* BEq*-consistent pair of sets of formulas, then there is a prime theory* $\Gamma \supseteq \Sigma$ *such that* $\Gamma \cap \Delta = \emptyset$*.*

The Belnap-Gabbay Extension Lemma is a standard tool in the construction of prime theories to be used in completeness proofs for relevant logics.

Next, we can proceed to define the canonical frame for BEq. The parts apart from S are standard.

- K is the set of all prime BEq-theories.
- $a \in N$ iff $\mathsf{BEq} \subseteq a$.
- $Rabc$ iff for all B, C, if $B \to C \in a$ and $B \in b$, then $C \in c$.
- $a^* = \{A : \neg A \notin a\}$.

[22]Details and proofs are provided by Routley et al. (1982), especially chapter 4, or Restall (2000), especially chapters 5 and 11.

To the preceding, we need to add the canonical accessibility relation, S. In order to define S, it will be useful to define an auxiliary concept.

- Given a set of formulas x, $\Box^{-}x = \{A \in \mathcal{L} : \Box A \in x\}$.

Now, we can define the modal accessibility relation S for the canonical frame.

- Sab if and only if (i) $\Box^{-}a \subseteq b$ and (ii) $\Box^{-}b \subseteq a$.

It remains to show that the canonical frame obeys the various conditions and that in the canonical model, truth at a world coincides with membership in that world. We note a standard lemma, whose proof we omit.

Lemma 19. *In the canonical frame, $a \leq b$ if and only if $a \subseteq b$.*

Next, we will show that S is an equivalence relation.

Lemma 20. *S is reflexive.*

Proof. This is immediate from $\mathsf{(T)}$ being a theorem of BEq. □

Lemma 21. *S is symmetric.*

Proof. This is immediate from the definition. □

Lemma 22. *S is transitive.*

Proof. Suppose that Sab and Sbc. We show that Sac. Let $A \in \Box^{-}a$. Then, by the $\mathsf{(4)}$ axiom, $\Box A \in \Box^{-}a$. So, $\Box A \in b$ and $A \in c$. The converse is proven in the same way, with an appeal to symmetry. □

Next, we will show that the canonical frame obeys condition (i) on being an equivalence frame.

Theorem 23. *If $a \leq a'$ and $Sa'b'$ there is some world b such that Sab and $b \leq b'$.*

Proof. Suppose that $a \leq a'$ and $Sa'b'$. Let Δ be the set of formulas D such that $D \notin b'$. And let Σ be the set of formulas $\Box S$ such that $S \notin a$. Then we know that

$$\Box^{-}a \nvdash \Delta \cup \Sigma.$$

For, if $\Box^{-}a \vdash \Delta \cup \Sigma$, then there would be $A_1, ..., A_n \in \Box^{-}a$, $D_1, ..., D_m \in \Delta$ and $\Box S_1, ..., \Box S_p$ such that

$$(1) \ \vdash (A_1 \wedge ... \wedge A_n) \to ((D_1 \vee ... \vee D_m) \vee (\Box S_1 \vee ... \vee \Box S_p))$$

But then, by regularity,

$$(2)\ \vdash \Box(A_1 \wedge ... \wedge A_n) \to \Box((D_1 \vee ... \vee D_m) \vee (\Box S_1 \vee ... \vee \Box S_p))$$

And, since

$$(3)\ \vdash (\Box S_1 \vee ... \vee \Box S_p) \to \Box(\Box S_1 \vee ... \vee \Box S_p)$$

we have

$$(4)\ \vdash \Box(A_1 \wedge ... \wedge A_n) \to \Box((D_1 \vee ... \vee D_m) \vee \Box(\Box S_1 \vee ... \vee \Box S_p)).$$

From (4) and axiom (M5),

$$(5)\ \vdash \Box(A_1 \wedge ... \wedge A_n) \to (\Box(D_1 \vee ... \vee D_m) \vee \Box(\Box S_1 \vee ... \vee \Box S_p)).$$

Thus, $\Box(D_1 \vee ... \vee D_m) \vee \Box(\Box S_1 \vee ... \vee \Box S_p) \in a$. But a is prime, so either $\Box(D_1 \vee ... \vee D_1) \in a$ or $\Box(\Box S_1 \vee ... \vee \Box S_p)$ is in a. If $\Box(D_1 \vee ... \vee D_m) \in a$, then $\Box(D_1 \vee ... \vee D_m) \in a'$ and so $D_1 \vee ... \vee D_m \in b$. b is also prime, and so one of the D_i is in b for some $1 \leq i \leq m$ and this is contrary to the construction of Δ. Similarly, if $\Box(\Box S_1 \vee ... \vee \Box S_p) \in a$, then, by (T), $\Box S_1 \vee ... \vee \Box S_p \in a$ and so by primeness and (T), for at least one of the S_i, $S_i \in a$, contrary to the construction of Σ. This completeness the reductio and shows that $\Box^{-}a \not\vdash \Delta \cup \Sigma$.

By the Belnap-Gabbay lemma, there is a prime theory b such that $\Box^{-}a \subseteq b$ and $b \cap (\Delta \cup \Sigma) = \emptyset$. We show that Sab. It suffices to show that $\Box^{-}b \subseteq a$. Suppose that $\Box A \in b$. Then $\Box A \notin \Sigma$. By the definition of Σ, $A \in a$. Generalising, $\Box^{-}b \subseteq a$.

Moreover, $b \cap \Delta = \emptyset$ and so, by the definition of Δ, $b \subseteq b'$. Hence, $b \leq b'$, ending the proof of the lemma. □

It is worth emphasizing that in our proof that the canonical frame obeyed condition (i), we used both (M5) and (T). We have not yet found a way to prove the preceding theorem without appeal to (T).

As is usual, we will define the canonical valuation V for the canonical model, as follows.

- $V(p) = \{a \in K : p \in a\}$.

That $V(p)$ is closed upwards under the heredity ordering is immediate from lemma 19. The canonical model is the canonical frame with the canonical valuation. It remains to prove the Truth Lemma, showing that in the canonical model, truth and membership coincide. There is only one new cases that we need to prove, namely the case for $\Box$.

Lemma 24. *For all worlds $a \in K$ and formulae A, $\Box A \in a$ if and only if for all $b \in K$ such that Sab, $b \in A$.*

Proof. $\Longrightarrow$ follows from the definition of S for the canonical model.

$\Longleftarrow$ Let a be an arbitrary world (i.e. prime theory). Suppose that $\Box B \notin a$. We construct a world b such that $B \notin b$ and Sab. Let Δ be the set of formulae $\Box D$ such that $D \notin a$. And let Δ' be the disjunctive closure of $\{B\} \cup \Delta$.

We show that $\Box^{-}a \nvdash \Delta'$. For if $\Box^{-}a \vdash \Delta'$ we would have

$$(1)\ \vdash (A_1 \wedge ... \wedge A_n) \rightarrow (B \vee (\Box D_1 \vee ... \vee \Box D_m)),$$

for some $A_1, ..., A_n \in \Box^{-}a$ and $\Box D_1, ..., \Box D_m \in \Delta$. By lemma 15 and (1),

$$(2)\ \vdash (A_1 \wedge ... \wedge A_n) \rightarrow (B \vee \Box(\Box D_1 \vee ... \vee \Box D_m)).$$

From (2) and (Mono) we have

$$(3)\ \vdash \Box(A_1 \wedge ... \wedge A_n) \rightarrow \Box(B \vee \Box(\Box D_1 \vee ... \vee \Box D_m)).$$

We now appeal to an instance of axiom (M5):

$$(4)\ \vdash \Box(B \vee \Box(\Box D_1 \vee ... \vee \Box D_m)) \rightarrow (\Box B \vee \Box(\Box D_1 \vee ... \vee \Box D_m)).$$

(3) and (4) give us

$$(5)\ \vdash \Box(A_1 \wedge ... \wedge A_n) \rightarrow (\Box B \vee \Box(\Box D_1 \vee ... \vee \Box D_m))$$

and an appeal to axiom (T) yields

$$(6)\ \vdash \Box(A_1 \wedge ... \wedge A_n) \rightarrow (\Box B \vee \Box D_1 \vee ... \vee \Box D_m).$$

It follows that $\Box B \vee \Box D_1 \vee ... \vee \Box D_m \in a$. But a is prime and $\Box D_i \notin a$ for $1 \leq i \leq m$. So, $\Box B \in a$ contrary to assumption. Thus, by reductio, we conclude that $\Box^{-}a \nvdash \Delta'$. In other words, $(\Box^{-}a, \Delta')$ is an BEq-consistent pair. By the Belnap-Gabbay Extension Lemma, we can extend $\Box^{-}a$ to a prime theory b such that Sab and $b \cap \Delta' = \emptyset$.

To verify that Sab, we need to show that $\Box^{-}a \subseteq b$ and $\Box^{-}b \subseteq a$. The former is immediate from the construction of b. For the latter, suppose that $C \in \Box^{-}b$, and suppose for reductio that $C \notin a$. From the former assumption it follows that $\Box C \in b$. From the latter assumption, it follows that $\Box C \in \Delta$, and so $\Box C \in \Delta'$. Therefore, $\Box C \in b \cap \Delta'$, contradicting $b \cap \Delta' = \emptyset$. Therefore, $C \in a$, as desired.

Since $B \in \Delta'$ and $b \cap \Delta' = \emptyset$, it follows that $B \notin b$. Therefore, we can conclude that we have the desired theory b such that Sab and $B \notin b$.

□

Lemma 25. *For all worlds $a \in K$ and formulae A, $A \in a$ if and only if $a \Vdash A$.*

Proof. The proof is by induction on the complexity of A. The preceding lemma handles the only novel case, namely the case when A is of the form $\Box B$. $\Box$

The pieces having been assembled, we can now state the completeness theorem with a sketch of the proof.

Theorem 26. *If A is valid in the class of all equivalence frames for* B*, then A is a theorem of* BEq.

Proof. Suppose A is not a theorem of BEq. Then $(\mathsf{BEq}, \{A\})$ is a BEq-consistent pair. In the canonical model, there is a world $a \in N$ such that $A \notin a$, which suffices to show that A is not valid in the class of all equivalence frames for B. $\Box$

It is worth noting that, although the rule (Nec), $A \Rightarrow \Box A$, does not hold for BEq, it can be added. If one does add (Nec), then one needs to add the frame condition that if Sab and $a \in N$, then $b \in N$. The resulting logic is sound and complete with respect to the resulting class of frames.

We will also note, without proof, that the completeness result offers an alternative route to showing that BEq enjoys the variable-sharing property, that if $A \to B$ is valid, then A and B share an atom. The results of Standefer (2025a) show that BEq enjoys the variable-sharing property by being a sublogic of a logic that has the variable-sharing property, as that property is preserved downwards to sublogics. Alternatively, the same proof method of theorem 4 of Standefer (2020) can be used to demonstrate directly that BEq enjoys the variable-sharing property.[23]

With completeness for BEq established, we will now turn to some other metatheoretic results.

5 Metacompleteness

In this section we will present a metacompleteness result for the logic $\mathsf{BEq.Nec}$, which is BEq extended with the rule (Nec).[24] Metavaluations are a tool introduced by Meyer (1971; 1976) that mix features of proof theory and algebras. Meyer's metavaluations only worked for stronger relevant logics, but they were generalized to weaker logics,

[23]For more on the importance of variable-sharing, see Standefer (2025b).

[24]We will note that our nomenclature for logics is less systematic than the useful dotting convention of Ferenz (2023). $\mathsf{BEq.Nec}$ extends BEq with a rule, while BEq extends B with several modal axioms and a rule. To be more systematic, one could write $\mathsf{B.Eq}$ and $\mathsf{B.EqNec}$, but, as we are not discussing a wide range of logics, we will stick with the present nomenclature.

including B, by Slaney (1984). Seki (2013) generalized Slaney's metavalutions to a range of relevant modal logics.[25] In this section, we will use metavaluations to obtain some results concerning a strengthening of BEq.

First we define metavaluations, following Slaney and Seki.

Definition 27. *A* metavaluation *for* L *is a pair of functions m and m^* from $\mathcal{L}$ to $\{0,1\}$ such that the following hold.*

- $m(p) = 0$, $m^*(p) = 1$, *for all* $p \in \mathsf{At}$.
- $m(\neg B) = 1$ *iff* $m^*(B) = 0$;
  $m^*(\neg B) = 1$ *iff* $m^*(B) = 0$.
- $m(B \wedge C) = 1$ *iff* $m(B) = 1$ *and* $m(C) = 1$;
  $m^*(B \wedge C) = 1$ *iff* $m^*(B) = 1$ *and* $m^*(C) = 1$.
- $m(B \vee C) = 1$ *iff* $m(B) = 1$ *or* $m(C) = 1$;
  $m^*(B \vee C) = 1$ *iff* $m^*(B) = 1$ *or* $m^*(C) = 1$.
- $m(B \to C) = 1$ *iff (i)* $\vdash_{\mathsf{L}} B \to C$, *(ii) if* $m(B) = 1$ *then* $m(C) = 1$, *and (iii) if* $m^*(B) = 1$, *then* $m^*(C) = 1$;
  $m^*(B \to C) = 1$.
- $m(\Box B) = 1$ *iff both* $\vdash_{\mathsf{L}} \Box B$ *and* $m(B) = 1$;
  $m^*(\Box B) = 1$ *iff* $m^*(B) = 1$.

A formula A holds on a metavaluation m, m^* *iff* $m(A) = 1$.

Throughout this section, L will be BEq.Nec. For those familiar with metavaluations, we are working with the clauses for M1t metavaluations.

Slaney showed that there is an important connection between metavaluations and provability, which was extended by Seki. We will present it here for BEq.Nec, although it was proved for other logics by Slaney and Seki.

Theorem 28. *For all formulas A and metvaluations m, m^*, the following are true.*

- *If* $m(A) = 1$, *then* $\vdash_{\mathsf{BEq.Nec}} A$.
- *If* $m^*(A) = 0$, *then* $\vdash_{\mathsf{BEq.Nec}} \neg A$.

Proof. These are proved by simultaneous induction on the complexity of A. The proof is essentially the same of that of Slaney and Seki. □

[25] Brady (2017) provides an excellent overview of work on metavaluations.

The main result concerning metavaluations is the following.

Theorem 29. *For all formulas A and metvaluations m, m^*, if $\vdash_{\mathsf{BEq.Nec}} A$, then $m(A) = 1$.*

Proof. This is proved by induction on the length of the proof of A. Most of the cases are covered by the work of Seki (2013). The only new case is for (M5).

The provability clause for m is satisfied, so we show that if $m(\Box(A \vee \Box B)) = 1$, then $m(\Box A \vee \Box B)) = 1$, and if $m^*(\Box(A \vee \Box B)) = 1$, then $m^*(\Box A \vee \Box B)) = 1$.

Suppose that $m(\Box(A \vee \Box B)) = 1$. It follows that $\vdash_{\mathsf{BEq.Nec}} \Box(A \vee \Box B)$ and $m((A \vee \Box B)) = 1$. This implies $m(A) = 1$ or $m(\Box B) = 1$. If $m(A) = 1$, then $\vdash_{\mathsf{BEq.Nec}} A$. By (Nec), this implies $\vdash_{\mathsf{BEq.Nec}} \Box A$, so $m(\Box A) = 1$. This suffices for $m(\Box A \vee \Box B) = 1$. Suppose that $m(\Box B) = 1$. This suffices for $m(\Box A \vee \Box B) = 1$. Therefore, $m(\Box A \vee \Box B) = 1$.

Next, suppose that $m^*(\Box(A \vee \Box B)) = 1$. Using the evaluation clause for $\Box$ with m^*, we have $m^*(\Box(A \vee \Box B)) = m^*(A \vee \Box B)$, and $m^*(A \vee \Box B) = 1$ iff $m^*(A) = 1$ or $m^*(\Box B) = 1$. Since $m^*(A) = m^*(\Box A)$, we can conclude $m^*(A \vee \Box B) = 1$ iff $m^*(\Box A \vee \Box B) = 1$. By assumption, $m^*(\Box(A \vee \Box B)) = 1$, so it follows that $m^*(\Box A \vee \Box B) = 1$, as desired.

□

As a corollary of the preceding two theorems, we obtain the disjunction property for BEq.Nec.

Corollary 30. *If $\vdash_{\mathsf{BEq.Nec}} A \vee B$, then either $\vdash_{\mathsf{BEq.Nec}} A$ or $\vdash_{\mathsf{BEq.Nec}} B$.*

Proof. Immediate from the preceding two theorems. □

From this we can obtain the admissibility of the rule form of disjunctive syllogism, aka γ.[26]

Corollary 31. *The rule $\neg A, A \vee B \Rightarrow B$ holds for* BEq.Nec.

Proof. Suppose $\vdash_{\mathsf{BEq.Nec}} \neg A$ and $\vdash_{\mathsf{BEq.Nec}} A \vee B$. From the previous corollary, it follows that either $\vdash_{\mathsf{BEq.Nec}} A$ or $\vdash_{\mathsf{BEq.Nec}} B$. BEq.Nec is consistent in the sense that it does not have $A \wedge \neg A$ as a theorem, for any formula A, as it is a sublogic of S5, which is consistent. Since $\vdash_{\mathsf{BEq.Nec}} \neg A$, it follows that $\nvdash_{\mathsf{BEq.Nec}} A$. Therefore, $\vdash_{\mathsf{BEq.Nec}} B$, as desired.

□

[26] The rule γ has a long history in the study of relevant logics. Meyer et al. (1984) and Øgaard (2019; 2021) provide critical discussion of γ in relevant logics. For some results concerning γ in relevant modal logics, see Mares and Meyer (1992) or Seki (2011; 2012).

These results do not immediately extend to BEq, as the provability claims of the consequents may fail in BEq. In the proof of theorem 29, we needed to appeal to (Nec) in one place, and, as far as we can tell, that appeal is essential for the proof using the metavaluational techniques. Showing that BEq has the disjunction property, if it indeed does, will have to proceed by other means.

6 Halldén Completeness

The previous section shows that BEq.Nec has the disjunction property, that is, if $A \vee B$ is a theorem of the logic, then either A is a theorem or B is a theorem. Our proof relies on the logic's being closed under the rule of necessitation, and so cannot be extended to BEq. In the present section, we show that BEq has a weaker property, that is, Halldén completeness.

Definition 32. *A logic* L *is Halldén complete if and only if, for any formulae A and B, if $\vdash_L A \vee B$, then at least one of A or B is a theorem of L or A and B share a propositional variable.*

The method we use to prove this is a modification of van Benthem and Humberstone's (1983) proof for modal logics based on classical logic.[27] The idea is to take two arbitrary non-theorems of BEq, A and B that do not share any variables, and show that we can find a model that falsifies $A \vee B$. We do so by gluing together a model invalidating A and one invalidating B. The model that results from the gluing is related to both of the original models by functions called "relevant pseudo-epimorphisms" or *rp-morphisms*. An rp-morphism for BEq frames is defined as follows:

Definition 33. *Given two* BEq *frames $\mathcal{F} = \langle K, N, R, S, ^* \rangle$ and $\mathcal{F}' = \langle K', N', R', S', ^{*\prime} \rangle$, an* rp-morphism *is a function f from K onto K' such that the following seven conditions hold:*

1. *if $a \in N$ then $f(a) \in N'$;*
2. *for all $a \in N'$ there is some $x \in N$, such that $f(x) = a$;*
3. *if $Rabc$ then $R'f(a)f(b)f(c)$;*
4. *if Sab then $Sf(a)f(b)$;*

[27] For the extension to other relevant modal logics, see Mares (2003) and Seki (2015).

5. *if $R'f(a)bc$ then there are $x, y \in K$, $Raxy$, $f(x) = b$, and $f(y) = c$;*

6. *if $S'f(a)b$ then there is an $x \in K$, Sax and $f(x) = b$;*

7. $f(a^*) = f(a)^*$.

Before we can prove a key theorem, we need to define one concept.

Definition 34. *An equivalence frame $\langle K, N, R, S, ^* \rangle$ is* R-serial *iff for all $a \in K$, there are $b, c \in K$ such that $Rabc$.*

Equivalence frames for B need not be R-serial. Nonetheless, restricting to R-serial frames does not change the logic. The reason is that the canonical frame is R-serial, in virtue of the trivial theory, the set of all formulas, being included in it.[28] This fact carries over from frames for B to equivalence frames. We will need to appeal to R-seriality in a key step of the next theorem.

Theorem 35. *Let $\mathcal{F}$ and $\mathcal{F}'$ be R-serial equivalence frames and f an rp-morpshism from $\mathcal{F}$ to $\mathcal{F}'$. Let V and V' be value assignments on $\mathcal{F}$ and $\mathcal{F}'$ respectively, such that for all propositional variables p, $a \in V(p)$ if and only if $f(a) \in V'(p)$. Then, for all formulae A, $a \Vdash A$ if and only if $f(a) \Vdash A$.*

Proof. By induction on the length of formulae.

Base Case. For all propositional variables $a \Vdash p$ if and only if $f(a) \Vdash p$ by the condition of the lemma.

Case. A is a conjunction, $B \wedge C$. By the inductive hypothesis, $a \Vdash B$ if and only if $f(a) \Vdash B$ and $a \Vdash C$ if and only if $f(a) \Vdash C$. So, by the truth condition for conjunction, $a \Vdash B \wedge C$ if and only if $f(a) \Vdash B \wedge C$.

Case. A is a negation, $\neg B$. Suppose that $a \Vdash \neg B$. Then $a^* \nVdash B$. By the inductive hypothesis, $f(a^*) \nVdash B$. By condition 7 of the definition of an rp-morphism, $f(a^*) = f(a)^*$. Hence $f(a)^* \nVdash B$ and so $f(a) \Vdash \neg B$.

Going the other way, suppose that $f(a) \Vdash \neg B$. Then $f(a)^* \nVdash B$, and so $f(a^*) \nVdash B$. By the inductive hypothesis, $a^* \nVdash B$ and so $a \Vdash \neg B$.

Case. A is an implication, $B \to C$. Suppose that $a \Vdash B \to C$. Then for all $b, c \in K$, if $Rabc$ and $b \Vdash B$, then $c \Vdash C$. By R-seriality, there are $x', y' \in K'$ such that $Rf(a)x'y'$. Then by condition 5 of the definition of an rp-morphism, there are $x, y \in K$ such that $Raxy$, $f(x) = x'$ and $f(y) = y'$. By the inductive hypothesis, if $x' \Vdash B$, then $y' \Vdash C$.

[28] Mares (2003) used the condition of total seriality, which added S-seriality to R-seriality. The equivalence frames all satisfy S-seriality, so we omit it here.

Now suppose that $f(a) \Vdash B \to C$. Then for all $b, c \in K'$, if $R'f(a)bc$ and $b \Vdash B$, then $c \Vdash C$. Assume that $Raxy$. By condition 3 of the definition of an rp-morphism, $Rf(a)f(x)f(y)$. By the inductive hypothesis, if $x \Vdash B$ then $y \Vdash C$ and so $a \Vdash B \to C$.

Case. A is a necessitive, $\Box B$. Suppose that $a \Vdash \Box B$. Then for all $b \in K$, if Sab then $b \Vdash B$. Suppose now that $S'f(a)x'$. Then, by condition 6 of the definition of an rp-morphism, Sax where $f(x) = x'$. By the inductive hypothesis, $x' \Vdash B$. Generalising, $f(a) \Vdash \Box B$.

Now suppose that $f(a) \Vdash \Box B$. Suppose also that Sax. By condition 4 of the definition of an rp-morphism, $Sf(a)f(x)$. By the truth condition for necessity, $f(x) \Vdash B$ and by the inductive hypothesis $x \Vdash B$. Generalising, $a \Vdash \Box B$. □

Now suppose that for some $a \in N'$, $a \nVdash A$. By condition 3 of the definition of an rp-morphism, there is some $x \in N$ such that $f(x) = a$. Thus, there is some $x \in N$ such that $x \nVdash A$. This shows that $\mathcal{M}$ invalidates all the fomulae that $\mathcal{M}'$ invalidates. But we are going to need more than this to prove Halldén completeness. In order to provide the further material we need, we follow van Benthem and Humberstone and define rp-fusions of BEq frames:

Definition 36. *Let $\mathcal{F}_1 = \langle K_1, N_1, R_1, S_1, {}^{*_1} \rangle$ and $\mathcal{F}_2 = \langle K_2, N_2, R_2, S_2, {}^{*_2} \rangle$ be* BEq *frames and $a_1 \in K_1$ and $a_2 \in K_2$. $(\mathcal{F}, a)$ is an* rp-fusion *of $(\mathcal{F}_1, a_1)$ and $(\mathcal{F}_2, a_2)$ if there are rp-morphisms f_1 and f_2 from $\mathcal{F}$ to $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively such that $f_1(a) = a_1$ and $f_2(a) = a_2$.*

For any two such $(\mathcal{F}_1, a_1)$ and $(\mathcal{F}_2, a_2)$, we can construct an rp-fusion. The constructed frame is just the product of $\mathcal{F}_1$ and $\mathcal{F}_2$: The product, $\mathcal{F}_1 \times \mathcal{F}_2$ is $\mathcal{F} = (K, N, R, S, *)$ where $K = \{(b_1, b_2) : \ b_1 \in K_1 \ \wedge \ b_2 \in K_2\}$, $N = \{(b_1, b_2) : \ b_1 \in N_1 \ \wedge \ b_2 \in N_2\}$, $R(b_1, b_2)(c_1, c_2)(d_1, d_2)$ if and only if both $R_1 b_1 c_1 d_1$ and $R_2 b_2 c_2 d_2$, $S(b_1, b_2)(c_1, c_2)$ if and only if both $S_1 b_1 c_1$ and $S_2 b_2 c_2$, and $(b_1, b_2)^* = (b_1^{*_1}, b_2^{*_2})$. Then the rp-fusion of $(\mathcal{F}_1, a_1)$ and $(\mathcal{F}_2, a_2)$ is $(\mathcal{F}, (a_1, a_2))$. The rp-morphisms f_1 and f_2 are projection functions, that is, $f_1(b_1, b_2) = b_1$ and $f_2(b_1, b_2) = b_2$. It is easy, although tedious, to prove that this product construction produces rp-fusions of BEq frames.

Now that we have rp-fusions, we can prove Halldén completeness.

Theorem 37. *The logic* BEq *enjoys Helldén completeness.*

Proof. Suppose that A and B are non-theorems of BEq that do not share any propositional variables. Then, by completeness with respect to R-serial frames, there is an R-serial frame $\mathcal{F}_1 = \langle K_1, N_1, R_1, S_1, {}^{*_1} \rangle$ and $a_1 \in N_1$ such that $a_1 \nVdash A$ and similarly an R-serial frame $\mathcal{F}_2 = \langle K_2, N_2, R_2, S_2, {}^{*_2} \rangle$ and $a_2 \in N_2$ such that $a_2 \nVdash B$. Let $\mathcal{F}$

be the product of $\mathcal{F}_1$ and $\mathcal{F}_2$. We construct a value assignment V on $\mathcal{F}$ such that for all variables p in A, $(b_1, b_2) \in V(p)$ if and only if $b_1 \Vdash p$, and for all q in B, $(b_1, b_2) \in V(q)$ if and only if $b_2 \Vdash q$. By theorem 35, $(a_1, a_2) \nVdash A$ and $(a_1, a_2) \nVdash B$. Hence, by the truth condition for disjunction, $(a_1, a_2) \nVdash A \vee B$. So, by soundness, $A \vee B$ is not a theorem of BEq. □

7 Possibility

We have not, so far, discussed possibility. As is common for work on relevant modal logics, we will define possibility as dual to necessity, namely $\Diamond A$ is $\neg\Box\neg A$. With this definition in mind, we can define a derived accessibility relation for $\Diamond$, $S_\Diamond$.

$$S_\Diamond ab \ =_{df} \ Sa^*b^*$$

The truth condition for $\Diamond$ comes out as the expected one using this relation, as proven in the following fact.

Fact 38. *For every world a, $a \Vdash \Diamond A$ if and only if there is some world b such that $S_\Diamond ab$ and $b \Vdash A$.*

Proof.

1.	$a \Vdash \Diamond A$	iff	$a \Vdash \neg\Box\neg A$
2.		iff	$a^* \nVdash \Box\neg A$
3.		iff	$\exists c \in K(Sa^*c \wedge c \nVdash \neg A)$
4.		iff	$\exists b \in K(Sa^*b^* \wedge b^* \nVdash \neg A)$
5.		iff	$\exists b \in K(S_\Diamond ab \wedge b \Vdash A)$

The equivalence between lines two and three holds due to the verification condition for $\Box$. The equivalence between the third and fourth lines holds because, every $b \in K$ is such that for some $c \in K$, $b = c^*$. □

From the definition, it follows that $S_\Diamond$ is an equivalence relation as well

Fact 39. *$S_\Diamond$ is an equivalence relation.*

Proof. 1. For reflexivity, let a be an arbitrary world. By definition, $S_\Diamond aa$ if and only if Sa^*a^*, which holds because S is reflexive.

2. For transitivity. suppose that $S_\Diamond ab$ and $S_\Diamond bc$. By definition, Sa^*b^* and Sb^*c^*, hence by the transitivity of S, Sa^*c^*. Therefore, by definition, $S_\Diamond ac$.

3. For symmetry, suppose that $S_\Diamond ab$. By definition, Sa^*b^*. Hence, by the symmetry of S, Sb^*a^* and so $S_\Diamond ba$. □

Thus, $S_\Diamond$ inherits many of the nice features of S. It is worth noting, however, that in general, Sab need not imply $S_\Diamond ab$, or the converse. Requiring that $S = S_\Diamond$ would result in coordinated equivalence frames, defined by Standefer (2025a).

Definition 40 (Coordinated equivalence frames)**.** Coordinated equivalence frames *are those equivalence frames where for all* $a, b \in K$*, if* $[a] = [b]$*, then* $[a^*] = [b^*]$*.*

We will now prove the indicated equivalence.

Fact 41. *Let* F *be an equivalence frame for* B*. Then the following are equivalent.*

(1) If $[a] = [b]$*, then* $[a^*] = [b^*]$*.*

(2) $S = S_\Diamond$*.*

Proof. For one direction, assume (1). Suppose that Sab, so $[a] = [b]$. By (1), $[a^*] = [b^*]$, so Sa^*b^*, which is to say $S_\Diamond ab$. Next, suppose $S_\Diamond ab$. This implies Sa^*b^*, which in turn implies $[a^*] = [b^*]$. From the assumption, this gives us $[a^{**}] = [b^{**}]$. Given that $c^{**} = c$, for all $c \in K$, $[a] = [b]$, so Sab, as desired.

For the other direction, assume (2) and suppose that $[a] = [b]$. By assumption, Sab, so $S_\Diamond ab$, by (2). This means that Sa^*b^*, so $[a^*] = [b^*]$, as desired. □

Requiring that $S = S_\Diamond$ significantly strengthens the logic BEq, as it results in both (B) and (5) both being valid, as shown by Standefer (2025a). As avoiding (B) and (5) is touted as one of the features of BEq, we will not pursue investigation of the coordinated equivalence frames here.[29]

$S_\Diamond$ can be viewed as representing the situations that are open epistemic possibilities, given the agent's information knows. In general, $S \neq S_\Diamond$, and we think that this may be useful when a belief operator, $\mathcal{B}$, is added to the language and interpreted using its own accessibility relation, $S_{\mathcal{B}}$. The relevant epistemic logician has the option of enforcing (or not) relations between belief and knowledge, as well as between belief and open epistemic possibilities. A natural next step is exploring a richer, multi-modal epistemic logic, as is common in classically-based epistemic logics, and the distinction between S and $S_\Diamond$ provides additional flexibility for modeling different epistemic scenarios.[30] We leave this idea for future work, as adequate investigation of multi-modal logics is outside the scope of the present paper.

[29]Standefer (2025a) proved a completeness result for the coordinated equivalence frames. While that result was proven in the context of R, it can be carried out for B as well.

[30]See, for example, Stalnaker (2006).

8 Conclusion

The main result of this paper is proving the completeness of BEq with respect to equivalence frames for B. In addition, we have shown that BEq enjoys Halldén completeness. A slightly stronger logic, BEq.Nec has the disjunction property. These features demonstrate that BEq and BEq.Nec are metatheoretically well-behaved logics. It is worth noting that the results can be extended to many logics stronger than B, although the metavaluational techniques are more sensitive to the choice of base logic.

We close with a lingering open question. We have provided a completeness proof for equivalence frames, where the accessibility relation is an equivalence relation. In particular, the relation is symmetric. In the present setting, the (B) axiom is insufficient for delivering the frame condition of symmetry in the canonical frame. Inspection of our proofs reveal that, given the definition of the canonical modal accessibility relation, we need to appeal to (M5), as well as (T). The open question is: are there some axioms that can either be used instead of (M5) or in addition to it that can be used to obtain a completeness result with respect to symmetric frames that are not necessarily reflexive or transitive? It would be good to have either a proof of completeness that did not require the (T) axiom or a proof that no such result is possible.

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