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Routes to relevance: Philosophies of relevant logics

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Abstract

Relevant logics are a family of non-classical logics characterized by the behavior of their implication connectives. Unlike some other non-classical logics, such as intuitionistic logic, there are multiple philosophical views motivating relevant logics. Further, different views seem to motivate different logics. In this article, we survey five major views motivating the adoption of relevant logics: Use Criterion, sufficiency, meaning containment, theory construction, and truthmaking. We highlight the philosophical differences as well as the different logics they support. We end with some questions for future research.Logic & Philosophy of Language

Relevant logics, in the tradition coming out of the work of Anderson and Belnap (1975), are concerned with implication. Relevant logics constitute a large family with great variety, even restricting attention to the comparatively well known logics.[1] Perhaps unsurprisingly, different philosophical motivations have been given for relevant logics, targeted at different subfamilies of the broader group.[2] In this article I will survey the different philosophical views motivating relevant logics, indicating how they secure relevance, what logics are most clearly supported by those views, and the presentation of the logic most naturally supported by the view.

1 | BACKGROUND

We will focus on propositional logic in the basic logical vocabulary with → (implication), ¬ (negation), ∧ (conjunction), and ∨ (disjunction).[3] For the purposes of this paper, we will primarily consider logics as sets of logical truths.[4]

Perhaps the easiest way to characterize relevant logics is negatively: They are characterized by the rejection of the paradoxes of implication, many of which were highlighted by C. I. Lewis. In the purely implicational fragment, this means rejection of the following two paradoxes of implication,

- $A \to (B \to A)$, and
- $B \to (A \to A)$,

both of which are valid in classical and intuitionistic logic. Once other connectives are considered, additional principles must be rejected in order to avoid letting the implicational paradoxes back in, such as $((A \wedge B) \to C) \to (A \to (B \to C))$ and $A \to (B \to (A \wedge B))$.[5,6]

While different relevant logicians disagree about many principles, there is substantial agreement, both on principles that should be accepted and on those that must be rejected. For example, $A \to A$ is generally regarded as alright, while $A \to (B \to A)$ is not. One thing that has universal support among relevant logicians, at least at the propositional level, is the use of Belnap's variable-sharing criterion as a necessary condition on being a relevant logic.[7]

> *Variable-sharing criterion*: A logic $\mathtt{L}$ satisfies the *variable-sharing criterion* iff for any formulas A, B, if $A \to B$ is valid in $\mathtt{L}$, then A and B share a propositional variable.[8]

The variable-sharing criterion, at the level of propositional logic, captures a sense of formal relevance for the implication connective. There is a formal connection between the antecedent and consequent, some shared non-logical content, witnessed by the shared propositional variable. The variable-sharing criterion requires rejecting both paradoxes of implication, along with whatever entails them. The second paradox immediately violates the criterion. To see that the first paradox also violates it, take the instance of the paradox $(p \to p) \to (q \to (p \to p))$. If $p \to p$ is also valid, then by *modus ponens* we get as valid $q \to (p \to p)$, which is an instance of the second paradox and a violation of the variable-sharing criterion.

Many logics used in philosophy, such as classical logic ($\mathtt{C}$) and intuitionstic logic, violate the variable-sharing criterion, so the criterion has some teeth. To illustrate, the second of the paradoxes of implication, $A \to (B \to B)$, is valid in intuitionistic logic, so it is not a relevant logic.[9] The criterion is standardly taken to be a necessary (but not sufficient) condition on being a relevant logic.[10] Beyond the core of agreement that implication should exhibit such a connection, there is much disagreement, both about which principles should be accepted and the philosophical views behind the logics.

There are many relevant logics that one can obtain from axioms and rules using the basic set of connectives, although most of the standard relevant logics are extensions of the basic relevant logic $\mathtt{B}$. To help orient the reader, we present in Table 1 the axioms and rules for the logic $\mathtt{B}$. In the table, the double arrow, $\Rightarrow$, is used to indicate a rule, where on the left are the premises of the rule and on the right is its conclusion. As an example, the first rule listed, $A, A \to B \Rightarrow B$, is *modus ponens*. To obtain stronger logics, we can add axioms or rules to $\mathtt{B}$. The logics extending $\mathtt{B}$ that come up in this paper are presented in Table 2, indicating how to obtain one logic from another by adding axioms. For example, to get the logic $\mathtt{TW}$, we add the prefixing, suffixing, and contraposition axioms to $\mathtt{B}$, and to get $\mathtt{T}$, we further add the contraction and reductio axioms to $\mathtt{TW}$. Each of the stronger logics keeps the rules and axioms of the logic(s) it is extending.[11] The relations among logics in this paper are presented in the diagram of Figure 1, where arrows indicate containment.[12]

The relevant logics will be divided into two groups, the *non-contractive* logics, among which are $\mathtt{B}$, $\mathtt{DJ}$, $\mathtt{TW}$, and $\mathtt{RW}$, and the *contractive* logics, among which are $\mathtt{T}$, $\mathtt{E}$, and $\mathtt{R}$. The distinction tracks whether the logics have contraction principles, such as $(A \to (A \to B)) \to (A \to B)$.[13] The differences among logics, apart from their grouping into contractive and non-contractive, do not matter much for the ensuing discussion. One important difference between the groups is that the contractive logics contain all the classical tautologies in the vocabulary $\{\wedge, \vee, \neg\}$,

TABLE 1 Axioms and rules for B.

Axioms	Rules
$A \to A$	$A, A \to B \Rightarrow B$
$(A \wedge B) \to A$, $(A \wedge B) \to B$	$A, B \Rightarrow A \wedge B$
$A \to (A \vee B)$, $B \to (A \vee B)$	$A \to B \Rightarrow (C \to A) \to (C \to B)$
$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$	$A \to B \Rightarrow (B \to C) \to (A \to C)$
$((B \to A) \wedge (C \to A)) \to ((B \vee C) \to A)$	$A \to \neg B \Rightarrow B \to \neg A$
$(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$	
$\neg\neg A \to A$, $A \to \neg\neg A$	

TABLE 2 Relevant logics extending B.

To get	From	Add
DJ	B	$((A \to B) \wedge (B \to C)) \to (A \to C)$ (conjunctive syllogism)
		$(A \to \neg B) \to (B \to \neg A)$ (contraposition)
TW	B	$(A \to B) \to ((C \to A) \to (C \to B))$ (prefixing)
		$(A \to B) \to ((B \to C) \to (A \to C))$ (suffixing)
		$(A \to \neg B) \to (B \to \neg A)$ (contraposition)
RW	TW	$(A \to (B \to C)) \to (B \to (A \to C))$ (permutation)
T	TW	$(A \to (A \to B)) \to (A \to B)$ (contraction)
		$(A \to \neg A) \to \neg A$ (reductio)
R	RW	$(A \to (A \to B)) \to (A \to B)$ (contraction)
E	T	$((A \to A) \to B) \to B$ (E axiom)
		$(\Box A \wedge \Box B) \to \Box(A \wedge B)$ (E axiom),
		where $\Box C$ is defined as $(C \to C) \to C$

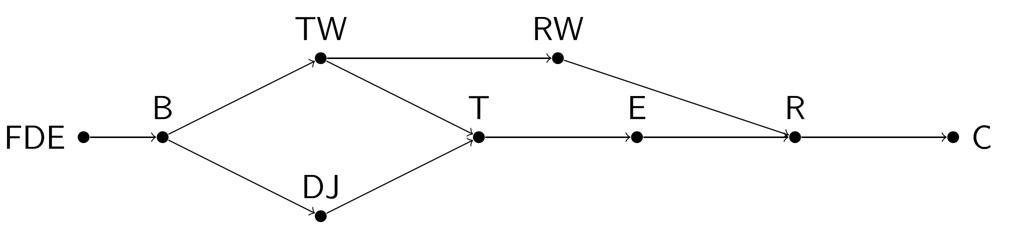

FIGURE 1 Relations of logics in this paper.

while the non-contractive logics do not. Thus, the contractive logics are, in a sense, closer to classical logic while the non-contractive ones are further away.

We will briefly note some differences between relevant logics and classical logic. In relevant logics, $A \to B$ is not equivalent to $\neg A \vee B$. Some classical tautologies that are not valid in any of the relevant logics discussed include $A \to (B \vee \neg B)$, $(B \wedge \neg B) \to A$, $((A \to B) \to A) \to A$, $(A \to B) \vee (B \to A)$, $A \to (B \to A)$, and $A \to (B \to B)$. The non-contractive relevant logics depart further from classical logic than the contractive ones, as none of the following classically valid formulas are valid in the non-contractive logics though they are valid in the contractive ones: $A \vee \neg A$, $\neg(A \wedge \neg A)$, $(A \wedge (A \to B)) \to B$, and $(A \to \neg A) \to \neg A$.

In the following sections, we will look at differing motivations for relevant logics and philosophical views offered in their support. The logics that will be mentioned have proof systems, different kinds of models, and soundness and completeness results linking them. Nonetheless, the different motivations make certain approaches to logic more natural, such as the Use Criterion favoring a proof-theoretic approach. In the discussion, we will highlight what approaches fit most naturally with the different views. As will emerge, the main motivations do not pin down a unique logic as *the* logic of relevance. We will return to this point in the conclusion. Now, let us turn to the first view motivating relevant logics, the Use Criterion, which we will discuss in more detail than the others because it is the historically most prominent and the most widely discussed.

2 | USE CRITERION

The first motivation for relevant logics is the Use Criterion, due to Anderson and Belnap. Anderson and Belnap criticized the violations of relevance found in classical and intuitionistic logics, one of which is exemplified in the following derivation of a paradox of implication.[14]

$$
\begin{array}{lllll}
1 & | & \underline{p} & & \text{hyp} \\
2 & | & | & \underline{q} & \text{hyp} \\
3 & | & | & q & \text{rep } 2 \\
4 & | & q \to q & & \to\text{I } 2-3 \\
5 & & p \to (q \to q) & & \to\text{I } 1-4
\end{array}
$$

To borrow from Anderson and Belnap,

> In this example we indeed proved $[q \to q]$, but, though our eyes tell us that we proved it *under* the hypothesis $[p]$, it is crashingly obvious that we did not prove it *from* $[p]$: the defect lies in the definition, which fails to take seriously the word 'from' in 'proof *from* hypotheses'. And this fact suggests a solution to the problem: we should devise a technique for keeping track of the steps used, and then allow application of the introduction rule only when A is relevant to B in the sense that A is *used* in arriving at B.[15]

This idea leads to the *Use Criterion* of relevant logics, glossed by Mares as the criterion "that the *premises really be used in the derivation of the conclusion*."[16] To implement the Use Criterion, Anderson and Belnap use subscripted formulas in a Fitch-style natural deduction system. A new hypothesis gets a singleton subscript containing a new numeral, and the rules are sensitive to the subscripts. For example, the implication rules are the following.

$\to$ **I** From a derivation of B_α under the hypothesis $A_{\{k\}}$ to infer $A \to B_{\alpha \setminus \{k\}}$, provided $k \in \alpha$.
$\to$ **E** From $A \to B_\beta$ and A_α to infer $B_{\alpha \cup \beta}$.

In addition to these, there is a rule of reiteration, reit, for moving a formula into a subproof or repeating it, and it does not alter subscripts. The idea behind the subscripts for the implication rules is that if an assumption, $A_{\{k\}}$ is used in obtaining B_α, then the subscript on B, α, will reflect this fact by containing k. The function of the subscripts is easiest to see with an example, so we will illustrate by attempting to repeat the above derivation with the subscripts.

$$
\begin{array}{lll}
1 & \quad p_{\{1\}} & \text{hyp} \\
2 & \quad\quad q_{\{2\}} & \text{hyp} \\
3 & \quad\quad q_{\{2\}} & \text{reit } 2 \\
4 & \quad q \rightarrow q_{\emptyset} & \rightarrow\text{I } 2-3 \\
5 & p \rightarrow (q \rightarrow q)_{???} & \rightarrow\text{I } 1-4\ ✗
\end{array}
$$

Line 4 is justified by the → I rule because 2, the subscript on the hyp step at line 2, is in the subscript on the formula at the end of the subproof ending at line 3. The conclusion of the rule occurs outside the subproof with an empty subscript, because the → I rule requires its conclusion to be {2}\{2}, which is the empty set. The paradoxical derivation is blocked at line 5. The restriction on → I rule is not satisfied, as indicated by ✗, since the numeral on the assumption p at line 1 is not found in the subscript on $q \rightarrow q$. This indicates that line 1 is not really used in the derivation of line 4, capturing the idea that the initial assumption of p did not contribute to proving $q \rightarrow q$. Therefore, the application of → I in the final line would be inappropriate, and there is no subscript for the formula of line 5, as indicated by the question marks.

Consider the classical derivation of the other paradox of implication, presented on the left.

$$
\begin{array}{lll}
1 & \quad p & \text{hyp} \\
2 & \quad\quad q & \text{hyp} \\
3 & \quad\quad p & \text{reit } 1 \\
4 & \quad q \rightarrow p & \rightarrow\text{I } 2-3 \\
5 & p \rightarrow (q \rightarrow p) & \rightarrow\text{I } 1-4
\end{array}
\qquad
\begin{array}{lll}
1 & \quad p_{\{1\}} & \text{hyp} \\
2 & \quad\quad q_{\{2\}} & \text{hyp} \\
3 & \quad\quad p_{\{1\}} & \text{reit } 1 \\
4 & \quad q \rightarrow p_{???} & \rightarrow\text{I } 2-3\ ✗ \\
5 & p \rightarrow (q \rightarrow p)_{???} & \rightarrow\text{I } 1-4\ ✗
\end{array}
$$

As before, the geometry of the Fitch-style proof misleads; the mere fact that we have p under the hypothesis q is no reason to think that the former was derived *from* the latter. The proof reproduced with subscripts is presented on the right. As with the previous paradox, the use of subscripts blocks this derivation, with inappropriate uses of → I marked by ✗, after which point there are not appropriate subscripts, as indicated by the question marks. The subscripting makes it clear that line 2 was not properly involved in the justification of line 3, otherwise the subscript on line 3 would contain 2.

As an illustration of a successful derivation, we have the following derivation of contraction.

$$
\begin{array}{lll}
1 & \quad p \rightarrow (p \rightarrow q)_{\{1\}} & \text{hyp} \\
2 & \quad\quad p_{\{2\}} & \text{hyp} \\
3 & \quad\quad p \rightarrow (p \rightarrow q)_{\{1\}} & \text{reit } 1 \\
4 & \quad\quad p \rightarrow q_{\{1,2\}} & \rightarrow\text{E } 2,3 \\
5 & \quad\quad q_{\{1,2\}} & \rightarrow\text{E } 2,4 \\
6 & \quad p \rightarrow q_{\{1\}} & \rightarrow\text{I } 2-5 \\
7 & (p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)_{\emptyset} & \rightarrow\text{I } 1-6
\end{array}
$$

In this derivation, at line 4, →E combines the subscripts of its two premises. Line 6 comes from a successful application of →I, as we really used the antecedent, p, in obtaining the consequent, q, as indicated by the presence of subscript on the assumption of p at line 2 in the subscript on q in line 5.

The concept of use in the Use Criterion is most at home in proof systems, particularly natural deduction and sequent systems. In other sorts of proof systems, such as Hilbert-style axiom systems and tableaux systems, the concept of use is perhaps less natural, and for models it is less natural still.[17] As indicated above, the use of indexing on formulas permits one to track use in natural deduction systems. In sequent systems, the appropriate sort of use can be codified by rejection of the structural rule of weakening.[18]

Use is a holistic concept that takes different forms with other connectives, as illustrated by the rules for conjunction, ∧.[19] The introduction rule for ∧, from A_α and B_α to infer $A \wedge B_\alpha$, requires the *same* assumptions are used in obtaining both A and B, which is represented by them having the same subscript, α, which is attached to $A \wedge B$.[20] This is required to prevent the reintroduction of the paradoxes of implication via a detour through conjunction.[21]

The Use Criterion is, at first blush, compelling. Anderson and Belnap took the Use Criterion to motivate some of the stronger logics, R, E, and T. Apart from Anderson and Belnap, others have adopted a form of the Use Criterion in motivating certain logics.[22] Mares (2004) adopts a form of the Use Criterion in his theory of situated inference, which supports the logic R.[23] Read (1988) appeals to the Use Criterion, especially in connection with the fusion connective, although he has a separate account of relevance in validity.

The Use Criterion has, nonetheless, had its detractors. Both Copeland (1980) and Meyer (1985) have objected that the Use Criterion does not rule out irrelevance without a prior conception of use. A looser conception of use can result in classical logic, rather than a relevant logic.[24] And, further, even with a stringent conception of use, the Use Criterion does not uniquely pick out R, as illustrated by Brady (1984b).[25] The proper response, I think, is to concede these points to Copeland and Meyer. The Use Criterion does not, on its own, distinguish a privileged relevant logic, as there are different ways of formulating use. To land in the family of relevant logics, one must adopt a more substantive conception of use than in a conception that results in classical logic, what Mares calls "real use".

The Use Criterion of Anderson and Belnap motivates relevant logics with the idea that if $A \to B$ is a logical truth, one needs to use the antecedent in a substantive way to obtain the consequent. The concept of use is typically spelled out in a proof system. Anderson and Belnap's conception of use naturally results in R, but one can put more stringent requirements on use in order to obtain different relevant logics.[26] For example, one might prohibit repeated uses of assumptions by requiring that in $\to E$, the subscripts be disjoint, which will block contraction principles and lead to the logics RW or TW. We can see how the Use Criterion secures the variable-sharing criterion via a strict conception of use: If A and B do not share a variable, one will not be able to really use A in obtaining B.

To summarize, the Use Criterion view says that valid implication reflects the fact that antecedents are used, in a substantive way, in obtaining consequents. It is naturally associated with proof-theoretic presentations of logics, especially natural deduction and sequent systems. The Use Criterion has been primarily used in support of some of the stronger relevant logics, such as R, E, and T. Despite this, the Use Criterion is compatible with a range of logics, depending on the sense of *use* adopted.

3 | SUFFICIENCY

The next motivation, sufficiency, for relevant logics has been offered by Sylvan and Plumwood.[27]

Routley et al. (1982, p. x) say that genuine implication "amounts to total sufficiency". A sufficiency implication is one according to which nothing apart from the antecedent is needed for establishing the consequent.[28] It is, in the words of Routley (2019, p. 8), "a go-anywhere notion, which is not limited by the fact that situation in which it operates is somehow classically incoherent." In particular, neither changing the accessible situations nor even changing the laws of logic should result in the truth of the antecedent failing to suffice for the truth of the consequent.[29] A sufficiency implication connective must avoid illegitimately appealing to unstated truths. Traditional enthymemes are typically viewed as invalid, omitting contingent premises, sufficiency prohibits the omission of necessary or even logical truths. The reason is that sufficiency implication must work even where the omitted logical truths fail, as they can in models for relevant logics.

Sufficiency is itself sufficient for relevance, although the latter is not a foundational concept. Sylvan says, "Relevance of consequence [sic] to antecedent, though a hallmark of an adequate implicational relation, is strictly a by-product of a good sufficiency notion; for if *B* has nothing to do with *A* then *A* can hardly be sufficient for *B*."[30] If *A* is not even formally relevant to *B*, in the sense that they share no atoms, then it will be possible to make *A* true while *B* is false, thereby ensuring that *A* is not sufficient for *B*. Formal results, such as the Ackermann property, Maksimova property, or depth relevance properties provide further detail on ways sufficiency can fail.[31]

The sufficiency idea is first presented, by Routley and Routley (1972), in the context of first-degree entailment (`FDE`), a logic in which there are no nested implications and every formula is of the form $A \rightarrow B$ where A and B contain only atoms and the connectives $\wedge$, $\vee$, and $\neg$. `FDE` is important in the context of relevant logics because it is contained in all the standard relevant logics.[32] The axioms and rules of `FDE` are presented in Table 3.[33] Sylvan and Plumwood argue that, in the context of `FDE`, sufficiency is equivalent to two related ideas, suppression-freedom and maximum variation.

The first, suppression-freedom, says that the implication of the logic must reject suppression principles. According to Routley et al. (1982, p. 141), "A suppression principle is one that suppresses information required to proceed generally from antecedent to consequents. With suppression principles antecedents are not (everywhere) sufficient on their own for consequents." An example of a suppression principle is the rule $A, (A \wedge B) \rightarrow C \Rightarrow B \rightarrow C$. According to this rule, the sufficiency of A and B together for C would yield the sufficiency of B for C, given the validity of A. Traditional enthymemes are one form of suppression, but there are other forms of suppression. Classical logic exhibits suppression, for example in the validity of $q \rightarrow (p \rightarrow p)$, whose unsuppressed form is $(q \wedge (p \rightarrow p)) \rightarrow (p \rightarrow p)$.

Once one moves beyond `FDE` to the full relevant logics, it is not clear that suppression-freedom is equivalent to sufficiency. Øgaard (2020) provides a natural way of formalizing suppression-freedom in terms of the two principles.[34]

> *Anti-Suppressive Principle*: For every formula A, there are formulas B and C such that $(A \wedge B) \rightarrow C$ is valid but $B \rightarrow C$ is not.
>
> *Joint Force Principle*: For every formula A, there are formulas B and C such that $(A \wedge B) \rightarrow C$ is valid while neither $A \rightarrow C$ nor $B \rightarrow C$ is.

Øgaard shows that these principles are not stronger than Belnap's variable-sharing criterion. Indeed, while there are some principles identified as suppression principles that result in violations of Belnap's variable-sharing

TABLE 3 Axioms and rules of `FDE`.

Axioms	Rules
$(A \wedge B) \rightarrow A$, $(A \wedge B) \rightarrow B$	$A \rightarrow B, A \rightarrow C \Rightarrow A \rightarrow (B \wedge C)$
$A \rightarrow (A \vee B)$, $B \rightarrow (A \vee B)$	$A \rightarrow C, B \rightarrow C \Rightarrow (A \vee B) \rightarrow C$
$\neg\neg A \rightarrow A$, $A \rightarrow \neg\neg A$	$A \rightarrow B \Rightarrow \neg B \rightarrow \neg A$
$(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$	$A \rightarrow B, B \rightarrow C \Rightarrow A \rightarrow C$

criterion, such as $((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$,[35] there are some principles that are so identified that are compatible with relevance in the formal sense, e.g. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$.

The second idea that Sylvan and Plumwood say is equivalent to sufficiency in the context of FDE is the principle of maximum variation: "for every proposition *B* which is not a consequence of *A* there is some [situation] which *A* is in to which *B* does not belong. Any violation of this maximum variation principle will allow suppression somewhere"[36] To ensure that nothing is suppressed in the antecedent, that an antecedent really is sufficient for the consequent, one must consider a maximally varying array of situations. This has a clear meaning in the context of FDE, although, as with suppression-freedom, it is less clear how it is to be understood in the context of the full logics.

Sufficiency, as a basis for relevant logics, is naturally associated with model theory, such as the ternary relational models developed by Sylvan and Meyer.[37] The early work by Sylvan and Plumwood introducing sufficiency focused on models, and models can be used to make sense of the laws of logic varying, as these models will have points where some logical truths fail.[38] Different situations may falsify different putative logical laws. The model-based approach fits naturally with the tendency of proponents of sufficiency to defend weaker logics, such as B and DJ, since placing fewer conditions on models results in weaker logics being validated.[39]

Although there is a natural affinity between sufficiency and models, proponents of sufficiency, such as Sylvan and Plumwood, do talk about sufficiency in connection with deducibility. Plumwood (2023) explicitly talks about sufficiency linked with the Use Criterion of Anderson and Belnap.

Proponents of sufficiency do not advocate for a single logic. Sufficiency understood in terms of models, appears to support the weaker, non-contractive relevant logics. Routley (2019) strongly suggests that in his discussion of the non-contractive logics, there called "ultramodal logics", and Mares (2019) argues that Sylvan's desiderata favor a weak, non-contractive relevant logic.[40]

The sufficiency idea has some shortcomings. One shortcoming is that it is unclear how one should understand sufficiency, or the principle of maximum variation, in the context of models for the full logics, rather than the restricted context of FDE. A second is that sufficiency builds in a commitment to the claim that no formula has a substantive connection of meaning with *every* other formula, as in the Joint Force and Anti-Suppressive Principles. From the classical logician's point of view, every formula is sufficient for a tautology, as the meaning of a tautology is empty. The classical logician may consider every classical valuation, or every classical possible world, but mere truth-functional sufficiency is not enough to establish sufficiency in the sense of Sylvan and Plumwood. The classical logician has, according to the sufficiency advocate, not considered the recalcitrant situations where the laws of logic fail and other impossible worlds.[41] The principle of maximum variation needs to be formally specified in a way that rules out classical truth-functional sufficiency and other inadequate forms of sufficiency.

To summarize, the sufficiency view says that valid implication is a matter of the antecedent being sufficient for the consequent, no matter what sort of situation one is in, even highly degenerate or otherwise impossible situations. It is naturally understood in terms of models. It motivates relevant logics because the connection it requires of the implication connective must be strong. It delivers the variable-sharing criterion as a consequence, since if *A* and *B* do not even share an atom, there will be a situation in which *A* holds while *B* does not, meaning that *A* is not sufficient for *B*. The view has been used primarily to motivate some of the weaker, non-contractive relevant logics, around B and DJ, although it does not clearly single out one as the preferred relevant logic.

4 | MEANING CONTAINMENT

The next motivation for relevant logics is that of meaning containment. Brady (2006) points out that Belnap's basic relevance criterion fails to privilege a single logic, which he takes as an indication that it is too weak as a principle of logic choice. The variable-sharing criterion is a principle concerning mere meaning *connection*, which permits too many options for logics. Instead, he opts for strengthening the connection to be one of meaning *containment*. Brady proposes that implications express meaning containments, where the content of the antecedent contains that of the consequent.[42]

The meaning containment idea is used to argue against many principles of the stronger relevant logics that feature nested implications. These principles do not generally reflect meaning containment, as the meanings of their consequents are, plausibly, not contained in the meanings of their antecedents, such as $p \rightarrow ((p \rightarrow q) \rightarrow q)$, which is a theorem of $\mathtt{R}$. Brady has defended a few different, related logics of meaning of containment as he has refined his view on meaning containment. These logics share the feature of being relatively weak, non-contractive relevant logics. The logic Brady has most recently defended, $\mathtt{MC}$, differs from the logic $\mathtt{DJ}$, defended by Brady (2006), in dropping the distribution axiom, $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$, which is included in standard relevant logics.[43]

Brady has developed content algebra semantics, a kind of algebraic semantics, to provide some formal details for his views on meaning.[44] On this view, atoms have contents and the content of logically complex formulas computed from the contents of their parts.[45] On Brady's view, in a valid implication, the content of the antecedent contains the content of the consequent. An important feature of the view is that the content of implications includes distinctive, implicational content, not generally found in the content of a non-implication.[46] The content of implications includes anything that can be established from them. This fact can be used to explain why, for example, $p \rightarrow (q \rightarrow q)$ is invalid, as an atom need not have any implicational content.

The variable-sharing criterion appears to be supported by the meaning containment view. One this view, all formulas have content, and no formula is guaranteed to have a trivial content. If A and B do not share an atom, then, in general, the content of A need not be contained in that of B. The content of A may, for example, deal with Mount Taranaki, while the content of B may deal with the truths of group theory. Brady (1984a) presents a strengthened variable-sharing criterion that fits naturally with the meaning containment view.[47]

The meaning containment view has two potential shortcomings. The first is that, while the view works well to justify the positive axioms, dealing with negation requires additional conceptual machinery. For this purpose Brady introduces *ranges*, which are dual to contents, to interpret negation. Although formally elegant, they seem less tightly connected to the motivating idea of meaning containment than content. The second is that the view is difficult to combine with some of the other connectives often discussed in the area, such as fusion. Indeed, Brady (2006, pp. 34, 35) argues in favor of rejecting fusion and some other connectives on the basis of concerns stemming from meaning containment.

To summarize, the meaning containment view says that a valid implication expresses the containment of the meaning of its consequent in its antecedent. It is naturally connected with an algebraic semantics. It motivates the variable-sharing criterion: if A and B do not share a variable, the meaning of B will, generally, fail to be contained in that of A. The view provides a motivation for some non-contractive relevant logics. The view highlights two particular logics, $\mathtt{DJ}$ and $\mathtt{MC}$, as most strongly supported by the view.

5 | THEORY CONSTRUCTION

An alternative route to relevant logics is offered by the view that logical entailment is the general operation of theory closure.[48] A theory is a set of formulas closed under some notion of logical entailment, represented by $\rightarrow$, which is to say that for a theory X, if $A \in X$ and A entails B, then $B \in X$.[49] The closure of a set of formulas is an instance of the general operation of applying one set of formulas to another. Given sets of formulas X and Y, one can consider the *application* of X to Y, $X \bullet Y$, which will be the set of formulas B such that $A \rightarrow B \in X$ and $A \in Y$, for some A. One obtains the logical closure of a set of formulas X by applying the logic, considered itself as a theory, to X. Importantly, the application of one theory to another may fail to extend, or even be contained in, the latter theory. In this context, theories are not required to contain every logical truth. This allows for the distinction between theories *containing* a logic, namely containing every validity of the logic, and theories *conforming* to a logic, which is the case for a theory X if whenever $A \in X$ and $A \rightarrow B$ is valid in the logic, then $B \in X$. The distinction disappears in the face of non-relevant logics. Such logics typically validate the principle $A \rightarrow T$, where T is any logical truth, so every theory will contain every logical truth. Theory construction is used both by Logan and by Mares to motivate relevant logics.

Logan (2020, 2022, 2024), developing ideas from Beall (2017, 2018), argues for a relevant logic via the nature of theories.[50] Logic is a universal theory-building toolbox, which can be used to construct theories of all sorts. The general logical features of theories are supposed to come via the concept of theory application. Since logic is to be a *universal* toolbox, as few assumptions are made about the concept of application as possible. On this view, valid implications represent general principles of the universal theory-building toolkit. Satisfying the variable-sharing criterion comes as a byproduct of the universality of logic, because it guarantees that there will be enough theories to provide a counterexample to any implication where the antecedent and consequent do not share a variable.

Taking a different approach, Mares (2024) is concerned with developing a logic of *entailment*, represented by the implication connective. Entailment connects to theory construction because a logic of entailment is used to elaborate the commitments of collections of axioms. This route to relevance starts with the lesson that avoiding the paradoxes of implication is essential for a logic of entailment to be adequate. Otherwise, the logic will be contribute its own content, not just elaborating the commitments of axioms. Mares motivates some plausible principles with considerations derived from the concept of entailment.

Both Mares and Logan use theory application to construct models, adding an additional relation of theory containment.[51] Logan's models yield the logic $\mathtt{B}$. Mares obtains a range of entailment logics by adopting plausible principles on the application operator. In particular, Mares argues that considerations of necessity in connection with entailment support adopting conditions on models that validate Anderson and Belnap's logic $\mathtt{E}$.

In summary, the theory construction motivation connects valid implication with general approaches to building theories. It leads fairly naturally to models, although they are different kinds of models than arose in the previous sections. Variable-sharing comes out as a consequence of the level of generality required of theory construction, which results in few constraints being placed on the theory application operator. Mares's view is more tightly connected to variable-sharing, as it arises from concerns over entailment, while Logan's view seems less closely wedded to variable-sharing, as it arises more as a by-product of the background view of theories. As with the other motivations, theory construction does not clearly lead to a unique logic. Proponents have singled out two logics, $\mathtt{B}$ and $\mathtt{E}$, as plausible candidates for the preferred logics on this view.

6 | TRUTHMAKING

Truthmakers supply yet another route to relevant logics. According to truthmaker theory, a true statement is true in virtue of some state of affairs obtaining. For example, the state of affairs in which Mount Taranaki is on the north island of New Zealand makes true the claim "Mount Taranaki is on the north island of New Zealand", while it is not true in virtue of the state of affairs of the whole of the planet Earth. The latter state, in a sense, suffices for the claim's truth, due to containing a state in virtue of which it is true. This is the difference between exact and inexact truthmaking. The former requires that a verifying state "must be relevant *as a whole* to the truth of the statement"[52] while inexact truthmaking simply requires a state to contain an exact truthmaker. Exact truthmaking brings with it an intuitive notion of relevance, since if two claims share exact truth makers, they bear some relevance to each other.[53]

Jago (2020) uses exact truthmaking and mereological fusion to provide an account of implication.[54] The idea behind these models is that an implication $A \rightarrow B$ is made true by a state x if, whenever a state y making true A is put together, or mereologically fused, with x, the result is a state $x \sqcup y$ making true B. Merelogical fusion is standardly commutative, $x \sqcup y = y \sqcup x$, associative, $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$, and idempotent, $x \sqcup x = x$. Focusing on the implicational fragment, the truthmaker models yield the logic $\mathtt{R}$. One can obtain models for weaker logics by dropping postulates on mereological fusion, although the philosophical view works best and offers the most support for R.[55] Indeed, one departs from the theory of mereological fusion if one drops or alters the standard postulates.

It is important that a state exactly making true a claim does not suffice for a larger state exactly making true that claim. One could have two states x and y that, respectively, exactly make true p and q while the mereological fusion of states x and y, $x \sqcup y$, fails to exactly make true p, even though x is a part of $x \sqcup y$. Thus, x fails to exactly make true $q \rightarrow p$. This feature ensures that the paradoxes of implication can be invalidated.

The truthmaker view builds in a kind of relevance in the relation of exact truthmaking. This supports the variable-sharing criterion, as if A and B do not share an atom, then there will be a state that exactly makes true the antecedent while yielding no exact truthmaker for the consequent.

In summary, the truthmaker view says that valid implications reflect an important relation between exact truthmakers for antecedents and those of consequents. It naturally yields models for relevant implication. Incorporating the other connectives requires some additional conceptual machinery, in particular the addition of a relation of refinement of states and the relation of compatibility between states. These relations are both philosophically appealing and plausible, and the whole view provides a philosophical motivation for the logic R.

7 | CONCLUSION

We have surveyed five major motivations for relevant logics: the Use Criterion, sufficiency, meaning containment, theory construction, and truthmaking. These have natural affinities with different presentations of logics. The Use Criterion fits neatly with proof theory, the sufficiency view with model theory, particularly the Routley-Meyer models, meaning containment with content algebras, and the theory construction and truthmaking views with versions of operational models.

These views have been associated with different logics. The Use Criterion and truthmaking views have been associated with the contractive relevant logics, sufficiency and meaning containment with the non-contractive relevant logics, and the theory construction view has been associated with logic from both groups. To what extent are these associations accident of history as opposed to important features of the philosophical views? Getting clearer on the formal implications of the views may help settle the matter.

As noted, many of the motivations fail to privilege a single logic as *the* logic of relevance. Some logicians, such as Mares and Meyer (2001) view this as a feature, while others, such as Brady (2017), view it as a bug. Can any of those philosophical views be supplemented or augmented in a way that privilege a unique logic, *the* logic of relevance? Should they be? To what extent are these views compatible with pluralism about logic?[56] These seem to be promising avenues to pursue.

As indicated above, some of the motivations can be combined. Some of these combinations are compelling and underexplored, such as the combinations of use and sufficiency suggested by Plumwood (2023). There is room for further development of these views, both on the philosophical and formal sides.

There are some philosophical views related to relevant logics that we have not covered but we will mention them here. In a series of publications, Tennant has developed core logic, a kind of relevant logic, where the focus is on a relevance constraint imposed at the sequent level, rather than at the level of the object language implication.[57] Avron (1990, 1992) motivates the logic RM, which is related to the logic R but lacks the variable-sharing property. Finally, many authors have used informational semantics and informational interpretations of consequence in support of relevant logics.[58]

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ENDNOTES

[1] See Dunn and Restall (2002), Bimbó (2007), and Mares (2020) for general overviews of the area.

[2] This is a point noted, albeit in a more critical way, by Burgess (1983, p. 45), "Routleyanism and Andersonianobelnapianism are so dissimilar that it is misleading to apply a single label 'relevantism' to both."

[3] There are other connectives commonly considered in the context of relevant logics, such as intensional disjunction, aka fission, $+$, omitted for lack of space. For further discussion of options for relevant conenctives, see Standefer (2022).

[4] For some discussion of relevant logics in the context of consequence relations, see Avron (2014), Øgaard (2022), or Badia et al. (2022).

[5] I use "principle" here as a neutral term for both axioms and rules.

[6] These both lead to the first implicational paradox, given that $(A \wedge B) \rightarrow A$ is a logical truth. The former yields the first paradox via *modus ponens*, and the latter yields the first paradox via a basic transitivity principle, inferring $A \rightarrow (B \rightarrow D)$ from $A \rightarrow (B \rightarrow C)$ and $C \rightarrow D$.

[7] See Anderson and Belnap (1975, pp. 252–254) for a proof that many standard relevant logics satisfy the variable-sharing criterion. For further discussion of the variable-sharing criterion and its refinements, see Brady (1984a), Robles and Méndez (2011, 2012) and Logan (2021).

[8] For discussion of the variable-sharing criterion in the presence of propositional constants, see Yang (2013) and Øgaard (2021).

[9] It is worth noting one sometimes misunderstood feature of the variable-sharing property. Whether an axiom, or axiom scheme, produces violations of variable-sharing may depend on the rest of the logic. The second paradox of implication is a violation on its own, while other principles, such as $A \rightarrow (A \rightarrow A)$, the mingle axiom, lead to violations of variable-sharing when added to $\mathtt{R}$ but not when added to $\mathtt{T}$, as shown by Méndez et al. (2012). Because of this, one cannot simply say that just because an implicational axiom has variables shared between antecedent and consequent, its addition will not lead to violations of variable-sharing.

[10] For discussion of the sufficiency of variable-sharing, see Standefer (in press).

[11] The resulting axiomatizations are redundant, but that is fine for our purposes. Brady (1984b) provides non-redundant axiomatizations.

[12] The logic $\mathtt{FDE}$ is presented in Table 3. It trades some axioms of $\mathtt{B}$ for rules, in addition to having an additional restriction to be described later. The diagram in Table 1 says that $\mathtt{DJ}$ is contained in $\mathtt{T}$ even though conjunctive syllogism was not explicitly added in Table 2. The reason is that the addition of contraction, along with the suffixing and prefixing axioms and *modus ponens*, is enough to derive conjunctive syllogism.

[13] For more on contraction principles, see Restall (1993), Bimbó (2006), Robles and Méndez (2014a), and Shapiro and Beall (2021), among others. The study of naive set theory is an important motivation for the rejection of contraction principles. For more about naive set theory, see, for example, Routley (2019) or Weber (2010, 2021, 2022).

[14] In this section, we will talk about proofs involving assumptions, although the interest is in the theorems obtainable after discharging all assumptions.

[15] Anderson and Belnap (1975, p. 18). Emphasis in the original. The immediate context of the quotation is a Hilbert-style axiomatic derivation, but the points carry over to Fitch-style natural deduction derivations.

[16] Mares (2004, p. 6), emphasis in the original.

[17] There are, of course, senses of *use* to be found in these latter proof systems. Anderson and Belnap (1975) explain what this amounts to for the implication fragments of some logics, McRobbie and Belnap (1979) explain a way of

understanding it in tableaux systems, and Makinson (2017, 2022) supplies an alternative way of understanding it in a tableaux system.

[18] The rule of weakening for sequent systems is $\frac{X \succ A}{X, B \succ A}$. See Restall (2014a) for more on the representation of sequent structural rules in natural deduction systems.

[19] See Mares (2010) for a discussion of the rules for other connectives.

[20] Another conjunction-like connective, fusion (∘), is used when the subscripts on A and B may potentially differ: from A_α and B_β to infer $A \circ B_{\alpha\cup\beta}$. The distinction between conjunction and fusion plays an important role in proof systems for relevant logics. For discussion, see Read (1988) and Slaney (1990).

[21] If one adopts the rule that from A_α and B_β one can infer $A \wedge B_{\alpha\cup\beta}$, for example, then the paradoxes reemerge. From assumptions of $A_{\{1\}}$ and $B_{\{2\}}$, one could obtain $A \wedge B_{\{1,2\}}$. One application of &E yields $B_{\{1,2\}}$, from which $A \to (B \to A)_\varnothing$ follows by two applications of → I.

[22] It is worth noting that Prawitz (1965, ch. 7) presents a proof system for a relevant logic based on a conception of strict use, although it is a proof system for the semilattice logic of Urquhart (1972b). Urquhart (1989) presents an alternative proof system for this logic and argues for it based on considerations of use.

[23] The recent view of Mares (2024) is, in a way, a hybrid that combines the theory construction view and the Use Criterion.

[24] As an example, if the conclusion of $\to E$ is B_γ where $\gamma \supseteq \alpha \cup \beta$, then one can derive both implicational paradoxes and land at classical logic.

[25] For additional discussion and criticism of the Use Criterion in the context of consequence relations for relevant logics, see Øgaard (in press).

[26] Anderson and Belnap provide some restrictions to the sense of use that result in different logics. Meyer and McRobbie (1982) present another way of tracking use, as does Mares (2021), whose systems are in many ways is similar to those of Slaney (1990).

[27] Richard Sylvan and Valerie Plumwood both had the surname Routley earlier in their lives. They later changed their surnames, respectively, to Sylvan and Plumwood. We will follow the conventions of the field and cite their published work under the published names, otherwise referring to them by their adopted names.

[28] Routley et al. (1982, p. 141) claim that this idea can be found in Aristotle's *Prior Analytics*, where he says that in a valid syllogism "no further term is required from without to make the consequence necessary."

[29] Cf. Routley (2019, pp. 7, 8).

[30] Routley (2019, pp. 8, 9). Sylvan and Plumwood talk about entailment or implication relations in some places and connectives in others.

[31] For the first two, see Anderson and Belnap (1975, pp. 119, 243). The third was introduced by Brady (1984a), and for further discussion, see Robles and Méndez (2014a, 2014b) or Logan (2021).

[32] See Omori and Wansing (2017) for an overview of FDE. FDE has been widely used in philosophy and logic. For example, Belnap (1977a, 1977b) argues that FDE is how a computer should reason about databases potentially containing inconsistent or incomplete information, and Camp (2002) argues that FDE is a good logic for analyzing confusion.

[33] FDE is often presented as a sequent system, rather than as an axiom system, as done here, but this presentation indicates how FDE can be contained in our other relevant logics.

[34] These principles are also formulated by Plumwood (2023). Øgaard considers strengthenings of these principles as well.

[35] To elaborate this claimed violation, the validity of $A \to (B \to A)$ follows from $(A \wedge B) \to A$, given the principle.

[36] Routley and Routley (1972, p. 341).

[37] The key feature of ternary relational models is a ternary relation R on a set of points K, used to interpret implication with the following condition.

$x \Vdash B \to C$ iff for all $y, z \in K$, if $Rxyz$ and $y \Vdash B$, then $z \Vdash C$.

As with Kripke models for modal logics, one can obtain models for different logics by placing conditions on R. The logics of this paper are all sound and complete with respect to different classes of ternary relational models. For discussion of the philosophical interpretation of these models, see Beall et al. (2012).

38 See Sandgren and Tanaka (2020) for discussion of kinds of logical impossibility.

39 In this respect, perhaps the algebraic semantics of Meyer and Routley (1972), the simplified semantics of Priest and Sylvan (1992), or the collection frames of Restall and Standefer (2023) make for even better candidates for establishing sufficiency.

40 Nolan (2018) argues that Sylvan's desiderata can be met by much stronger logics.

41 Some classical logicians, such as Berto and Jago (2013), will admit impossible worlds into their models. Such logicians will have a different conception of what it is for an implication to hold in a model. One can admit incomplete situations into one's models and still get classical logic and even some of its modal extensions, as demonstrated by Humberstone (1981), for example.

42 It should be noted that Brady prefers the term 'entailments' for 'implications'. There are so-called logics of analytic implication, developed by Parry and Angell, for example, that also adopt a version of this principle. For discussion of connections between these approaches and relevant logics, see Deutsch (1985). See Ferguson (2017), French (2017), and Szmuc (2021) for some recent developments of these logics. Further exploration between logics of analytic implication and relevant logics seems promising.

43 See Brady and Meinander (2013).

44 See Brady (2006, ch. 2), and the citations found therein.

45 The contents described here are not sets of worlds. In some respects, they are similar to the topics discussed by Berto (2022).

46 See Brady (1996, 2006) for the updated content semantics with distinctive content for the implications.

47 This strengthened criterion, depth relevance, concerns sharing a proposition variable under the same number of nested implications. It is further discussed by Robles and Méndez (2014a, 2014b) and by Logan (2021).

48 A closure operator $\mathfrak{c}$ is defined to obey the following three conditions, where X and Y are sets: (i) $X \subseteq \mathfrak{c}(X)$, (ii) $\mathfrak{c}(X) = \mathfrak{c}(\mathfrak{c}(X))$, and (iii) if $X \subseteq Y$, then $\mathfrak{c}(X) \subseteq \mathfrak{c}(Y)$.

49 In the relevant logic context, there is typically a requirement that theories be closed under conjunction introduction, so that if $A \in X$ and $B \in X$, then $A \wedge B \in X$. We do not need to worry about this detail here.

50 Beall takes his arguments to support $\mathtt{FDE}$ rather than $\mathtt{B}$.

51 The models are similar to the operational models of Fine (1974) and Urquhart (1972a), rather than the ternary relational models of Routley and Meyer (1972a, 1972b). In these models, implications are interpreted using application as an operation on a set of theories K with the following condition.

$$x \Vdash B \rightarrow C \text{ iff for all } y \in K, \text{ if } y \Vdash B, \text{ then } x \bullet y \Vdash C.$$

52 Fine (2017, p. 628).

53 Van Fraassen (1969) demonstrated a natural connection between truthmakers, which he called facts, and $\mathtt{FDE}$. Restall (1996) arrives at a non-relevant, near relative of $\mathtt{FDE}$ through considerations of truthmakers as well.

54 The models based on truthmakers are formally similar to the operational models of Fine (1974) and of Urquhart (1972a). In particular, they use the following semantic clause.

$$x \Vdash B \rightarrow C \text{ iff for all } y \in K, \text{ if } y \Vdash B, \text{ then } x \sqcup y \Vdash C.$$

55 Majer et al. (2023) generalize the formal theory beyond $\mathtt{R}$ to weaker logics.

56 See Beall and Restall (2005), Restall (2014b), and Kouri (2016) for discussion of logical pluralism and relevant logics.

57 See Tennant (2017) for a presentation of the current form of core logic. Tennant (2015) compares core logic and $\mathtt{R}$ with respect to various variable-sharing properties.

58 See Urquhart (1972a), Slaney (1990), and Allo and Mares (2011), among others.

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