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ORIGINAL RESEARCH

On the hyperintensionality of relevant logics and some of their rivals

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Abstract
In this article, we present a definition of hyperintensionality appropriate to relevant logics. We then show that relevant logics are hyperintensional in this sense, drawing consequences for other non-classical logics, including HYPE and some substructural logics. We further prove some positive and negative results concerning extensionality and hyperintensionality in relevant logics. We close by discussing related concepts for classifying formula contexts and potential applications of these results in the area of epistemic logic.

Keywords Relevant logics · Hyperintensionality · HYPE · Extensionality

1 Introduction

Hyperintensionality, being able to distinguish necessarily equivalent formulas, has become an important topic in philosophical logic.[1] The growing importance of hyperintensionality for philosophical concepts has been highlighted by Nolan (2014), calling it the "hyperintensional revolution." One can, of course, extend classical logic with hyperintensional operators, but one might wonder whether other logics could offer something distinctive with respect to hyperintensionality.[2]

[1] See Berto and Nolan (2021).

[2] Some of the standard examples of hyperintensional operators added to classical logic, often though not always modeled using impossible worlds, include belief operators, knowledge operators, and conditional operators. See Wansing (1990), Alechina and Logan (2010), and Berto et al., 2018, among others, for recent examples, and see Berto and Jago (2019, ch. 7) for an overview of the work on epistemic logics. For a general approach to hyperintensional operators, see Sedlár (2019).

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Relevant logics provide examples of logics where there are surprises with respect to hyperintensionality. Relevant logics are a family of non-classical logics that can be used to draw fine-grained distinctions, including distinguishing logical truths. As we will see, many modal extensions of relevant logics exhibit hyperintensionality, and this includes extensions with standard axioms and rules. As an example, adding a standard assortment of S5 axioms and rules to a relevant logic will result in a modal logic with hyperintensional contexts, whereas adding those to classical logic does not yield a logic with such contexts.[3]

While our focus is on extensions of the logics with modal operators, there is a tradition in relevant logics of using the resources of the base logic to define a kind of necessity operator. The most prominent example of this is the logic E of entailment, defended by Anderson and Belnap (1975). One can ask whether the defined necessity operators exhibit any sort of hyperintensionality. We will argue that they do, providing a sense in which hyperintensionality is built into the base relevant logics. In so doing, we will draw out some consequences for other non-classical and substructural logics.

There is one additional place where discussion of hyperintensionality in non-classical contexts has arisen. Recently, Leitgeb (2019) defended the non-classical logic HYPE as exhibiting a distinctive combination of simplicity and strength. Among its claimed features is providing a kind of hyperintensionality, a claim disputed by Odintsov and Wansing (2021), who demonstrate that there is a sense of hyperintensionality, related to congruentiality below, that the logic does not enjoy. We will offer some support to Leitgeb's claim, proceeding via a discussion of relevant logics, showing that there is a sense in which HYPE is hyperintensional.

In the next section, we will supply some brief background on relevant logics, in particular the logic R. Then, in §3 we will precisely define some concepts to classify formula contexts, notably extensionality and hyperintensionality. In §4, we will present our main results concerning hyperintensional contexts in relevant logics. Then, in §5, we will make some observations concerning the concept of extensionality, drawing out a consequence for HYPE, and we will obtain some limitative results. Finally, in §6, we will look at two further definitions for classifying formula contexts and discuss some features of relevant logics reminiscent of hyperintensionality but distinct from it. We will close by discussing some potential upshots of our results for epistemic logics.[4]

[3] For concreteness, we mean (K) $\Box(A \to B) \to (\Box A \to \Box B)$, (4) $\Box A \to \Box\Box A$, (B) $A \to \Box\neg\Box\neg A$, $(\wedge\Box)$ $(\Box A \wedge \Box B) \to \Box(A \wedge B)$, (Nec) $A \Rightarrow \Box A$, although as we will see, many alternative sets of principles from S5 will work equally well for the present point. While $(\wedge\Box)$ is not typically taken as an axiom in S5, it is a theorem of the logic. It is included here since it is a standard axiom for modal relevant logics and to reassure the reader there is no funny business going on with the selection of modal principles.

[4] We note that this is an extended version of (Standefer, 2023a).

2 Relevant logics

Relevant logics are a family of non-classical logics with a distinctive conditional, or implication, connective.[5] One of the important ways in which the relevant conditional is distinctive can be found in Belnap's variable-sharing criterion: If $A \to B$ is valid, then A and B share a propositional variable. The variable-sharing criterion is typically taken as a necessary condition on being a relevant logic.[6] We will focus on the standard logical vocabulary of $\{\to, \wedge, \vee, \neg\}$, considering the addition of a modal operator $\Box$, below. The biconditional, $A \leftrightarrow B$, will be defined as $(A \to B) \wedge (B \to A)$. To contrast the relevant conditional and biconditional with the classical material ones, we will use $\supset$ and $\equiv$ for the latter connectives, defining $A \supset B$ as $\neg A \vee B$ and $A \equiv B$ as $(A \supset B) \wedge (B \supset A)$. In the context of relevant logics, and generally any non-classical logic, $A \supset B$ and $A \equiv B$ will be defined as in classical logic.

While there are many relevant logics, our focus will mostly be on the logic R, although we will look at a few others as well. R is a relatively strong logic.[7] We will present the axioms and rules for R, where $\Rightarrow$ is used to demarcate premises from conclusion in the rules.

(1) $A \to A$
(2) $(A \wedge B) \to A$, $(A \wedge B) \to A$
(3) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
(4) $A \to (A \vee B)$, $B \to (A \vee B)$
(5) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
(6) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
(7) $\neg\neg A \to A$
(8) $(A \to \neg B) \to (B \to \neg A)$
(9) $(A \to B) \to ((B \to C) \to (A \to C))$
(10) $A \to ((A \to B) \to B)$
(11) $(A \to (A \to B)) \to (A \to B)$
(12) $A, A \to B \Rightarrow B$
(13) $A, B \Rightarrow A \wedge B$

The logic R is the least set of formulas containing all the axioms and closed under the rules. Other relevant logics can be obtained by variation of axioms $(8) - (11)$, dropping those axioms or possibly adding others, and by addition of other rules. The focus will be on R, although we will briefly consider some weaker relevant logics towards the end of §4 and afterwards. The logic E can be obtained from R by changing axiom (10) to its rule form, $A \Rightarrow (A \to B) \to B$ and adding a reductio axiom, $(A \to \neg A) \to \neg A$. The logic T can be obtained from E by dropping the rule $A \Rightarrow (A \to B) \to B$. The logic {RW can be obtained from R by dropping (11). The

[5] See Dunn and Restall (2002), Bimbó (2007), Mares (2022), or Logan (2024) for overviews of the area. See Anderson and Belnap (1975) and Routley et al. (1982) for broader discussions.

[6] See Standefer (2025a) for discussion of the variable-sharing criterion as sufficient for being a relevant logic.

[7] See Mares (2004) for defense of R.

logic B, the weakest standardly considered relevant logic, can be obtained from R by dropping axioms $(8) - (11)$ and adding the following rules.

- $A \to \neg B \Rightarrow B \to \neg A$
- $A \to B \Rightarrow (C \to A) \to (C \to B)$
- $A \to B \Rightarrow (B \to C) \to (A \to C)$

There are other relevant logics one can consider, but we need not survey them here.[8]

Finally, we will consider logics as sets of logical truths.[9] Where A is a logical truth, or theorem, of L, we will write, $\vdash_{\mathsf{L}} A$. Let us now turn to some concepts for classifying formula contexts.

3 Classifying contexts

Let us begin with some definitions. Following Williamson (2006), define a formula context as a pair (C,p), of a formula and an atom. Given a context (C,p), the formula $C(A)$ is what results by replacing every occurrence of p in C with the formula A. We can then define an initial concept, extensionality for classical logic, CL, as follows.

Definition 1 *(Extensionality). A formula context (C,p) is extensional iff for all formulas A and B,*

- $\vdash_{CL} (A \equiv B) \supset (C(A) \equiv C(B))$.

This is a fine definition of extensionality for classical logic and its extensions. It is not, however, appropriate for all non-classical logics. The reason is that in many non-classical logics, including relevant logics, the interest is focused on the primitive conditional connective, and the associated biconditional, rather than the material conditional of the logic, and the associated material biconditional.[10] Therefore, we will replace the definition of extensional context with one that uses the appropriate conditional and biconditional of the logic.

Definition 2 *(Extensionality in L). A formula context (C,p) is extensional in the logic L iff for all formulas A and B,*

- $\vdash_{\mathsf{L}} (A \leftrightarrow B) \to (C(A) \leftrightarrow C(B))$.

[8] See Brady (1984) or Standefer (2025b), among others, for some axiomatizations of common relevant logics.

[9] This is the framework FMLA of Humberstone (2011).

[10] In the context of relevant logics, many of the contraction-free logics lack any theorems not containing '$\to$', for which see Slaney (1984); so $(\supset, \equiv)$-extensionality will be a less useful concept there. Yet, it still seems sensible to say that those logics have some extensional contexts made up only of the vocabulary $\{\wedge, \vee, \neg\}$. Thanks to an anonymous referee for raising this point.

This is a natural adaptation of Williamson's definition to non-classical logics. For a more general study of extensionality and related concepts, we would need to make the relativity to the chosen conditional and biconditional explicit, so that the two options above would be $(\supset, \equiv)$-extensionality and $(\rightarrow, \leftrightarrow)$-extensionality, respectively. There are alternative definitions of extensionality using different combinations of $\rightarrow$, $\supset$, $\leftrightarrow$, and $\equiv$, or even other conditionals and biconditionals, but we won't explore those further here.[11]

Extensional contexts are those that do not draw a distinction between equivalent formulas. Our interest here is not on extensional contexts per se, although we will return to them later in the paper. Our interest is, rather, in a related definition, that involves necessity, namely that of *hyperintensional* contexts.

To explain hyperintensional contexts, we will enrich the language with a necessity operator, $\Box$, to which we will return below. With necessity in the language, we can define hyperintensional contexts, although it will be worthwhile to define a related concept, intensional contexts. An intensional context is one where the necessary equivalence of formulas suffices for necessary equivalence in that context, but the mere equivalence of formulas does not. Following Williamson (2006), we first define an auxiliary concept and then define intensional contexts.

Definition 3 *Non-hyperintensionality in* L, *intensionality in* L. *A formula context (C, p) is non-hyperintensional in* L *iff for all formulas A and B,*

- $\vdash_L \Box(A \leftrightarrow B) \rightarrow \Box(C(A) \leftrightarrow C(B))$.

A formula context (C, p) is intensional in L *iff it is non-hyperintensional in* L. *and not extensional in* L.

In the last 50 years, interest in intensional pheonomena, and relatedly in intensional contexts, has grown in philosophy and philosophical logic, a trend noted by Nolan (2014), among others. A paradigm example of an intensional context would be $(\Box p, p)$ in the logic S5. This is not an extensional context, meaning that $S5(p \leftrightarrow q) \rightarrow (\Box p \leftrightarrow \Box q)$, but it is also not hyperintensional, meaning that for all A and B, $\vdash_{S5} \Box(A \leftrightarrow B) \rightarrow \Box(\Box A \leftrightarrow \Box B)$. It is worth noting that the use of a non-truth-functional connective does not suffice for contexts failing to be extensional, as shown by Humberstone (1986). While there is surely more to say about intensional contexts in non-classical logics, further investigation will be left for future work. Instead, let us finally turn to hyperintensional contexts, which will be the focus of most of this paper.

A hyperintensional context is one that "draws a distinction between necessarily equivalent" formulas.[12] In other words, a hyperintensional context is one where necessarily equivalent formulas may not necessarily be equivalent in that context.[13]

[11] See Humberstone (1986, 1997) and (Humberstone, 2011, 455) for more on extensionality of connectives.

[12] Berto and Nolan (2021).

[13] This is essentially the characterization given by Nolan (2014).

We will define hyperintensionality precisely, modeling our definition on that of Williamson (2006).

Definition 4 *(Hyperintensionality in* L*). A formula context (C,p) is hyperintensional in* L *iff there are formulas A and B such that*

- $L\Box(A \leftrightarrow B) \to \Box(C(A) \leftrightarrow C(B))$.

A logic L *is hyperintensional iff there is a formula context (C,p) that is hyperintensional in* L.

We will refer to the conditional, $\Box(A \leftrightarrow B) \to \Box(C(A) \leftrightarrow C(B))$*, as the hyperintensionality scheme.*

Before proceeding, we will make three comments on this definition. First, a context is hyperintensional in L iff it is not non-hyperintensional in L. Given that the focus of this paper is on hyperintensionality, it seems worthwhile to define the concept directly, rather than merely going via another definition. Second, although we are focused, in this paper, on some logics defined via axiom systems, the definition of hyperintensionality is given in a presentation-neutral way that applies equally well to logics defined in terms of proof systems, in terms of classes of frames, or via another method. Third, the definition is a straightforward generalization of a concept from classically-based modal logic, i.e. modal logics extending classical logic. As with other concepts moved from a classical setting to a non-classical one, there may be alternative, non-equivalent definitions that exhibit different features. This definition, however, seems appropriate since it uses the primary conditional and biconditional of the logic L. At least for the logics under discussion in this paper, the defined biconditional is the standard one, so it seems unlikely that an alternative definition of hyperintenisonality will offer an improvement.

Next, we will say that a logic M is a *sublogic* of a logic L iff M $\subseteq$ L. A consequence of the definitions so far is the following proposition.

Proposition 1 *Let* M *be a sublogic of* L*. If* L *is hyperintensional, then so is* M.

Proof Suppose that $L\Box(A \leftrightarrow B) \to \Box(C(A) \leftrightarrow C(B))$, for some A, B, and C. Since M $\subseteq$ L, $M\Box(A \leftrightarrow B) \to \Box(C(A) \leftrightarrow C(B))$, as desired. □

A context being hyperintensional is a failure of logical truth, or provability as appropriate, so hyperintensionality is preserved downwards to sublogics. This will be important for our main result. As we will be interested in demonstrating certain contexts, and logics, are hyperintensional, adopting the strongest logic will lead to the strongest result.

4 Hyperintensionality

In this section, we will prove some results concerning hyperintensionality for some modal extensions of the logic R. The definition of hyperintensional contexts should be understood as indexed to R, and its modal extensions, with the displayed conditional and biconditional being those of R. We will explicitly indicate when different logics are under consideration.

Once we have settled the question of the base logic, there is a further question concerning which necessity to use in the statement of hyperintensionality. For a general study of hyperintensionality, care needs to be taken regarding what modal axioms, if any, should be required to ensure that the hyperintensionality definition yields satisfactory results. Williamson uses universal necessity, which in the setting of classical logic is equivalent to the necessity of S5, in stating his definition.[14] The necessity of S5, or rather an S5-type extension of R, would be a fine necessity for our purposes, but we can obtain stronger results with a different necessity. A logic being hyperintensional is a matter of the invalidity of an instance of the hyperintensionality scheme, and, since invalidity is preserved from stronger logics down to weaker logics, using stronger modal principles will give stronger results concerning hyperintensionality.

Our aim will be to show that many plausible modal extensions of R are hyperintensional. To that end, we will consider the modal axiom scheme $A \leftrightarrow \Box A$, which is known as the TRIV axiom scheme. Let the logic R.TRIV be R with the addition of the TRIV axiom scheme. While R.TRIV is not a plausible modal logic for alethic necessity, it will work for our purposes.

To obtain our main result, we first prove a lemma using matrix methods. A matrix has a set V of semantic values, with a subset of designated values $D \subseteq V$, and operations on V for interpreting each connective of the language. A valuation v is a function from atoms to V that is extending to the whole language using the operations of the matrix. A valuation v on a matrix is a counterexample to a formula A iff $v(A) \notin D$.

Lemma 1 *The formula* $(p \leftrightarrow q) \rightarrow ((p \wedge r) \leftrightarrow (q \wedge r))$ *is not a theorem of R.*

Proof We will use a three-valued matrix. For the set of values, V, we take $\{0, \frac{1}{2}, 1\}$, with $D = \{\frac{1}{2}, 1\}$. The value of complex formulas is computed using the following tables. A valuation v is a countermodel for a formula A iff $v(A) = 0$, which is to say that $v(A)$ is not designated.

[14] The concept of S5-type necessity exhibits some subtleties in the context of relevant logics, for which see Standefer (2023b), and other non-classical logics, for an example of which see Ono (1977). In the present context, the addition of universal necessity to R would not result in a relevant logic, so we will not follow Williamson exactly.

$\to$	0	$\frac{1}{2}$	1	$\neg$
0	1	1	1	1
$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$
1	0	0	1	0

$\wedge$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	1	1	1

Every axiom of R is designated on every valuation and the rules preserve designation.[15] By an inductive argument, this implies that every theorem of R receives a designated value. To show that a formula is not a theorem of R, it suffices to provide a valuation that assigns it 0. In the case of interest, $v(p) = 1$, $v(q) = 1$, and $v(r) = \frac{1}{2}$ will work.[16] This valuation gives $v(p \leftrightarrow q) = 1$, while $v((p \wedge r) \leftrightarrow (q \wedge r)) = \frac{1}{2}$. As $1 \to \frac{1}{2} = 0$,

$$v((p \leftrightarrow q) \to ((p \wedge r) \leftrightarrow (q \wedge r))) = 0,$$

as desired. □

The formula scheme $(A \leftrightarrow B) \to ((A \wedge C) \leftrightarrow (B \wedge C))$ is not a theorem of R.[17] Next, we note a fact about R.TRIV.

Lemma 2 *Let (C, p) be a formula context. Then,* $\vdash_{\text{R.TRIV}} C(A) \leftrightarrow C(\Box A)$.

Proof The proof is by induction on the construction of C. □

With these lemmas in hand, we can turn to our main result.

Theorem 1 *The logic* $R.TRIV$ *is hyperintensional.*

Proof To show that R.TRIV is hyperintensional, we need a formula context which is hyperintensional. Take the formula context $(s \wedge r, s)$. The formula

$$\Box(p \leftrightarrow q) \to \Box((p \wedge r) \leftrightarrow (q \wedge r))$$

is not provable in R.TRIV. This is because we can use the previous lemma to focus on the equivalent

$$(p \leftrightarrow q) \to ((p \wedge r) \leftrightarrow (q \wedge r)),$$

which was shown not to be a theorem of R in lemma 1. □

[15] This was shown by Robert Meyer. See Anderson and Belnap (1975, 470).

[16] This countermodel was found using John Slaney's program MaGIC. See https://users.cecs.anu.edu.au/%7jks/magic.html

[17] Axioms of a similar form were studied by Routley (1982, 345) and by Urbas and Sylvan (1989), and these will be discussed more in the next section. Thanks to Andrew Tedder for drawing my attention to these citations.

Thus, we have demonstrated that R.TRIV is hyperintensional. It is worth noting that, for similar reasons, $(p \vee r, p)$ is a hyperintensional context as well. As an immediate corollary, we have the following result.

Corollary 1 *Let* L *be any sublogic of* $R.TRIV$. *Then* L *is also hyperintensional.*

The sublogics of $R.TRIV$ include all the well-known relevant logics, such as T, E, and B, as well as (multiplicative, additive) linear logic, and further it includes many of their extensions with well-known modal principles. We can sharpen this claim, but first we need a lemma.

Lemma 3 *Let* L *be a sublogic of* R*, and let* M *be an extension of* L *with modal rules and axioms. Suppose every theorem* A *of* M *has the feature that removing all occurrences of* $\Box$ *results in a theorem of* R*. Then,* M *is a sublogic of* R.TRIV.

Proof Suppose L is a sublogic of R, and let M be an extension of L with modal rules and axioms. Suppose every theorem A of M has the feature that removing all occurrences of $\Box$ results in a theorem of R. Let B be a theorem of M, and let C be the result of removing all occurrences of $\Box$. By assumption, C is a theorem of R. By repeated application of lemma 2, we can insert occurrences of $\Box$, obtaining a theorem of R.TRIV. $\Box$

As a consequence of the lemma and the preceding corollary, all logics satisfying the hypotheses will be hyperintensional. As an illustration of what these results cover, we note that all the relevant modal logics discussed by Fuhrmann (1990) fall within their scope, as do almost all relevant modal logics considered by Ferenz and Tedder (2022).[18]

There are modal logics that are not sublogics of TRIV, although the majority of the philosophically significant ones are sublogics of TRIV. Perhaps the most prominent modal logics that are not sublogics of TRIV are provability logics, logics that include the axiom $\Box(\Box A \to A) \to \Box A$.[19] These have not been studied much in the context of relevant logics, although RGL, a provability logic extension of R introduced by Mares (2000), provides an exception. Although the above countermodel does not work for Mares's provability logic, the same invalid formula demonstrates that the logic is hyperintensional.

Let us consider one further logic that is not a sublogic of R.TRIV. For this, we recall the three-valued matrix used in the proof of lemma 1. We add to that matrix an interpretation of $\Box$.

$$\Box x = \begin{cases} 1 & x = 1 \\ 0 & else \end{cases}$$

[18] The sole exception from Ferenz & Tedder (2022) is the logic RGL, to be mentioned below.

[19] See Boolos (1993) and Verbrugge (2017) for more on provability logics.

Let us say that the modal logic RM3U is the set of formulas that have no counterexamples on any valuation on this matrix, writing $\vdash_{\mathsf{RM3U}} A$ where A is a logical truth of RM3U.[20]

The modal logic RM3U is not a relevant logic, as $\vdash_{\mathsf{RM3U}} (p \wedge \neg p) \to (q \vee \neg q)$, which is a violation of the variable-sharing criterion.[21] It also lacks an axiom of TRIV, as $RM3Up \to \Box p$. The logic RM3U is not a great candidate for a logic of alethic necessity, but it contains some logics that are independently interesting. One such logic is the logic of universal necessity over R, presented by Standefer (2023b).[22] A full definition of this logic would require introducing ternary relational models and describing their relationship to matrices.[23] Rather than present those details, we will focus on RM3U and demonstrate its hyperintensionality.

Theorem 2 *The logic RM3U has hyperintensional contexts.*

Proof We will show that the context $(p \wedge r, p)$ is hyperintensional in RM3U. Consider the valuation v such that $v(p) = v(q) = 1$ and $v(r) = \frac{1}{2}$. It follows that $v(\Box(p \leftrightarrow q)) = 1$ while $v((p \wedge r) \leftrightarrow (q \wedge r)) = \frac{1}{2}$. Since $\Box\frac{1}{2} = 0$ it follows that

$$v(\Box(p \leftrightarrow q) \to \Box((p \wedge r) \leftrightarrow (q \wedge r))) = 0.$$

It then follows that $(p \wedge r, p)$ is hyperintensional in RM3U.

RM3U is hyperintensional, and it follows that all sublogics of RM3U are hyperintensional as well.[24] The logic RM3U falls outside the scope of theorem 1, so we can see that hyperintensionality arises for modal logics that do not satisfy the variable-sharing criterion, which we will also see in the next section with HYPE. For other modal logics that are not sublogics of R.TRIV, there is a lingering question of whether they are hyperintensional or not. In sect §6, we will show that many of those modal logics are hyperintensional, provided that they satisfy the variable-sharing criterion.

With theorem 1, we can specify a sense in which the base relevant logics are hyperintensional. This requires some additional background, and for the remainder of the section, we will remove $\Box$ from the language. Anderson and Belnap showed how to define logical necessity in their logic E, a close relative of R. Anderson and Belnap define $\blacksquare A$ as $(A \to A) \to A$.[25] This can be understood as saying that logic

[20] The name is based on two things. First, RM3 is a standard name for the logic obtained from this matrix, which is a three-valued extension of the logic RM, discussed in *Entailment volume 1* (Anderson & Belnap, 1975, §29), as well as elsewhere. Second, the U is because of the extension with the modal operator related to universal necessity discussed below.

[21] See the discussion of Anderson and Belnap (1975, 393ff). or Øgaard (2023).

[22] See Standefer (2022) for additional discussion.

[23] See Restall (2000, ch. 11) for a good introduction to ternary relational models. For the connections between ternary relational models, or rather frames, and matrices, the interested reader should consult *Relevant Logics and their Rivals vol. II* (Brady, 2003, ch. 9) or Standefer (2025b, ch. 5).

[24] Standefer (2023a, 246) claimed that the logic of universal necessity did not have hyperintensional contexts. Theorem 2 demonstrates that this was incorrect.

[25] One can obtain an alternative definition by using the Ackermann truth constant, *t*, which can be given an informal gloss as the conjunction of all logical truths. Using the Ackermann constant, $\blacksquare A$ can be defined

implies A, which is a fair definition of logical necessity. In the context of E, $\blacksquare$, so defined, has an S4-type logic, and in the context of weaker relevant logics, it obeys weaker principles. In the context of R, however, the defined connective $\blacksquare$ is trivial in the sense that $A \leftrightarrow \blacksquare A$ is a logical truth. Therefore, the context $(p \wedge r, p)$ is hyper-intensional, taking $\blacksquare$ to be the necessity of the hyperintsionality scheme, which is to say that $R\blacksquare(p \leftrightarrow q) \rightarrow \blacksquare((p \wedge r) \leftrightarrow (q \wedge r))$.

The defined necessity, $\blacksquare$, in the logic R obeys the TRIV principles. For weaker base logics, the defined necessity is weaker. For example, the defined necessity in the logic E does not obey the TRIV principles, but it does obey $\blacksquare A \rightarrow \blacksquare\blacksquare A$ and $\blacksquare A \rightarrow A$, among others. Using the defined necessity, we can view relevant logics as themselves (relevant) modal logics and use the defined necessity in the definition of hyperintensionality.[26] In this sense, R and its sublogics are hyperintensional. In particular, Anderson and Belnap's logic E, with its defined logical necessity operator, is hyperintensional. Since E with its defined necessity is simply E, it is natural to say that E is hyperintensional on its own. The sublogics of R build in hyperintensionality with respect to their defined necessity operators. Of course, there are other necessity operators one might define using the resources of the base logic. For many of these, the sublogics of R will be hyperintensional in much the same way.

With the main results on hyperintensionality proven, we will turn to a short discussion of extensionality, in light of the results above.

5 Extensionality and some limitative results

We will begin by observing one additional corollary of lemma 1.

Corollary 2 *There are contexts that fail to be extensional in* R.

Proof By lemma 1, $(s \wedge r, s)$ fails to be extensional in R. □

For similar reasons, $(s \vee r, s)$ also fails to be extensional in R. While it is perhaps not surprising that R, and all of its sublogics, contain non-extensional contexts, it is worth noting that the particular non-extensional contexts provided involve only conjunction or only disjunction, both often thought of as extensional.[27]

In the context of R, at least, Williamson's definition of extensional context, with $\supset$ and $\equiv$, would say that $(s \wedge r, s)$ is an extensional context, an $(\supset, \equiv)$-extensional context in the nomenclature of Sect. 3. This is not the case for many of the weaker relevant logics, a fact which is a consequence of the results of Slaney (1984). Many of the weaker relevant logics do not have any theorems that lack implications, and

as $t \rightarrow A$. The equivalence of the two definitions is demonstrated by Mares and Standefer (2017), among others.

[26] Viewing E as incorporating a modal element in its conditional was a point emphasized by Anderson and Belnap (1975), as can be seen from the subtitle to *Entailment*, namely "The logic of relevance and necessity."

[27] Cf. Gabbay (1978) corollary 21.

the $(\supset, \equiv)$-extensionality scheme does not add any arrows not contained in A, B, or C. Therefore, in such logics, $(\supset, \equiv)$-extensionality will not be an interesting or useful concept.

One might wonder whether there is an extension of R, or any of its sublogics, in which $\wedge$ and $\vee$ generate extensional contexts. We can provide a negative answer to this. Let us say that B.Ext is the logic obtained by B by adding the axioms

Ext1 $(A \leftrightarrow B) \rightarrow ((A \wedge C) \leftrightarrow (B \wedge C))$, and
Ext2 $(A \leftrightarrow B) \rightarrow ((A \vee C) \leftrightarrow (B \vee C))$.

These axioms may remind the reader of the (Factor) axioms studied by Urbas and Sylvan (1989):

F1 $(A \rightarrow B) \rightarrow ((A \wedge C) \rightarrow (B \wedge C))$, and
F2 $(A \rightarrow B) \rightarrow ((A \vee C) \rightarrow (B \vee C))$.

The (Factor) axioms are known to cause problems for relevant logics, in particular leading to violations of variable-sharing. A similar issue arises with the weaker extensionality axioms. We begin by proving a lemma.

Lemma 4 *Let (D, p) be a context built from the vocabulary* $\{\wedge, \vee\}$ *and containing an occurrence of p. Then* $(A \leftrightarrow B) \rightarrow (D(A) \leftrightarrow D(B))$ *is derivable in* B.Ext.

Proof The proof is by induction on the construction of D. If D is the atom p, then the result is immediate.

Suppose D is $E \wedge F$. There are two subcases: p occurs in both E and F or p occurs in only one. For the first subcase, by the inductive hypothesis, both $(A \leftrightarrow B) \rightarrow (E(A) \leftrightarrow E(B))$ and $(A \leftrightarrow B) \rightarrow (F(A) \leftrightarrow F(B))$ are derivable. By some straightforward reasoning, it follows that $(A \leftrightarrow B) \rightarrow ((E(A) \wedge F(A)) \leftrightarrow (E(B) \wedge F(B)))$ is derivable.

For the second subcase, without loss of generality, we can assume that p occurs only in E. By the inductive hypothesis, $(A \leftrightarrow B) \rightarrow (E(A) \leftrightarrow E(B))$ is derivable. Using (Ext1), it follows that $(A \leftrightarrow B) \rightarrow ((E(A) \wedge F) \leftrightarrow (E(B) \wedge F))$ is derivable.

The case where D is $E \vee F$ is similar, except that (Ext2) is used in the second subcase. □

As a corollary, we have the following.

Corollary 3 *The formula* $(p \leftrightarrow p) \rightarrow (((p \wedge q) \vee q) \leftrightarrow ((p \wedge q) \vee q))$ *is derivable in* $B.Ext$.

As a lemma, we will note that absorption is derivable in B.

Lemma 5 *In* B, $((A \wedge B) \vee B) \leftrightarrow B$ *is derivable.*

Combining this lemma with the preceding corollary gives the desired negative result.

Corollary 4 *The logic $B.Ext$ contains violations of variable-sharing.*

Proof Both $(p \leftrightarrow p) \to (((p \wedge q) \vee q) \leftrightarrow ((p \wedge q) \vee q))$ and $((p \wedge q) \vee q) \leftrightarrow q$ are derivable. Using some simple transitivity moves, it follows that $(p \leftrightarrow p) \to (q \leftrightarrow q)$ is derivable. □

As can be seen from the proofs, this result used very little as far as logical resources. Therefore, the hope of obtaining extensions of the standard relevant logics in which $\{\wedge, \vee\}$ generate extensional contexts is unsatisfiable. Although the extensionality axioms, (Ext1) and (Ext2), are weaker than the (Factor) axioms, (F1) and (F2), they give rise to what are, essentially, the same problems. Let us turn to the other connectives in relevant logics.

There is another binary connective, fusion ($\circ$), that is often considered in relevant logics.[28] The set of connectives $\{\to, \neg, \circ\}$ is sometimes informally described as intensional, or non-extensional. If we consider the contexts built from these connectives, we find that they are all extensional in R.

Proposition 2 *Let (C, p) be a context built from atoms and only the connectives $\to$, $\neg$, and $\circ$. Then, if p occurs in C, (C, p) is extensional in* R.

Proof The connective $\circ$ is definable in R as $A \circ B =_{Df} \neg(A \to \neg B)$. The result is then proved by induction on structure of *C*, which is straightforward using axioms (8) and (11). The inductive hypothesis is that $\vdash_{\mathsf{R}} (A \leftrightarrow B) \to (D(A) \leftrightarrow D(B))$, for less complex contexts (*D*, *p*).

For the conditional case, the context is $(D \to E, p)$. As $(D(A) \to E(A)) \to (D(A) \to E(A))$ is provable by axiom (1), we can prove

$$(A \leftrightarrow B) \to ((A \leftrightarrow B) \to ((D(A) \to E(A)) \to (D(B) \to E(B))))$$

with the two appeals to the inductive hypothesis and some simple transitivity moves available in R. An appeal to axiom (11) then yields half of the desired result. The other half is obtained similarly.

For the negation cases, we use (8) and the desired result follows immediately. □

Without the caveat that the *p* occurs in *C*, there can be a failure of extensionality for reasons of variable-sharing. Let *C* be *r*. Then (*r*, *p*) will be a failure of extensionality, as $(p \leftrightarrow q) \to (r \leftrightarrow r)$ would violate variable-sharing and so is not a theorem.

For logics that lack axioms (8), (10), or (11), the analog of proposition 2 may fail. In weaker logics, some contexts built from the connectives $\{\to, \neg, \circ\}$ can fail to be extensional. All the standard relevant logics include the rule form of axiom (9) used in the proof, so we will not consider dropping it here.

[28] See Read (1988) for a sustained discussion and defense of fusion.

Let us look at some examples of failures of extensionality in logics lacking axioms (8), (10), or (11). We start with the logic RW.[29]

Proposition 3 *In* RW*, the context* $(r \to r, r)$ *is not extensional.*

Proof We leave it to the reader to find a countermodel using MaGIC.

Next we will consider the logic T.[30] In T, fusion is not definable in terms of negation and conditional. Contexts built from fusion fail to be extensional.

Proposition 4 *In* T*,* $(p \circ r, p)$ *is not extensional.*

Proof We leave it to the reader to find a countermodel using MaGIC.

Although contexts built from fusion can fail to be extensional, in T, many contexts built from the vocabulary $\{\neg, \to\}$ are still extensional, as in R.

Proposition 5 *In* T*, all contexts (C, p) constructed from the vocabulary* $\{\to, \neg\}$ *and in which p occurs in C are extensional.*

Proof The negation and conditional cases from the proof from proposition 2 can be reproduced here, omitting fusion.

It is worth looking at an example of a failure of extensionality for contexts built from negation that can be found in the logic B.[31] Some formula contexts in the basic vocabulary fail to be extensional in B, beyond the examples provided above.

Lemma 6 *In* B*, the formula context* $(\neg p, p)$ *is not extensional.*

Proof In B,

$$(p \leftrightarrow q) \to (\neg p \leftrightarrow \neg q)$$

is invalid. We can adapt the matrix from the proof of lemma 1 to show this. We change the set of designated values to $\{1\}$, replace the conditional table with the following table and all valuations on the resulting matrix assign all the theorems of B designated values.[32] The valuation v where $v(p) = 1$ and $v(q) = \frac{1}{2}$ is a counterexample to the target formula.

[29] RW can be obtained from the axiomatization of R by dropping axiom (11).

[30] The logic T can be obtained from the axiomatization of R by removing (10) and adding $(A \to \neg A) \to \neg A$.

[31] The logic B can be obtained from R by dropping axioms $(8) - -(11)$ and adding the rules $A \to \neg B \Rightarrow B \to \neg A$, $A \to B \Rightarrow (C \to A) \to (C \to B)$, and $A \to B \Rightarrow (B \to C) \to (A \to C)$.

[32] This countermodel was found using John Slaney's program MaGIC. □

$\to$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	1	1
1	0	$\frac{1}{2}$	1

To obtain HYPE, or at least its logical truths, from R, we add $A \to (B \to A)$ and trade axiom (8) for its rule form, $A \to \neg B \Rightarrow B \to \neg A$. It follows that we can obtain HYPE by adding some axioms to B. B shares with HYPE the feature of having contraposition as a rule but, crucially, *not* as an axiom, which results in the failure of the pertinent instance of the extensionality scheme above. In fact, this example extends to HYPE as well. This provides an example of hyperintensionality in B.TRIV, as the context $(\neg p, p)$ is also hyperintensional in B.TRIV, and so in all sublogics. A similar point holds for HYPE, and in fact, the same matrix demonstrates the failure of extensionality. Thus, HYPE, and any sublogic of HYPE.TRIV also exhibits hyperintensionality in the same sense as relevant logics.

We have shown that it is not possible to extend a relevant logic at least as strong as B so that contexts built from $\{\wedge, \vee\}$ are extensional while maintaining variable-sharing. One might wonder whether it is possible to extend R, or indeed any relevant logic, with a necessity operator where the connectives $\{\wedge, \vee\}$ do not generate hyperintensional contexts while maintaining variable-sharing. We will now show that this is not possible.

For our result, we will add the axioms

Int1 $\Box(A \leftrightarrow B) \to \Box((A \wedge C) \leftrightarrow (B \wedge C))$, and
Int2 $\Box(A \leftrightarrow B) \to \Box((A \vee C) \leftrightarrow (B \vee C))$

as well as the rule

- (Cong), $A \leftrightarrow B \Rightarrow \Box A \leftrightarrow \Box B$,

to B. Let us call this logic B.Int.

Lemma 7 *In* B.Int*, if (D, p) is any context in the vocabulary* $\{\wedge, \vee\}$ *in which p occurs in D, then* $\Box(A \leftrightarrow B) \to \Box(D(A) \leftrightarrow D(B))$ *is derivable.*

Proof The proof is by induction on the construction of D. For the base case, $\Box(A \leftrightarrow B) \to \Box(A \leftrightarrow B)$ is an axiom

Suppose D is of the form $E \wedge F$. Without loss of generality, suppose that p occurs in E. By the inductive hypothesis, $\Box(A \leftrightarrow B) \to \Box(E(A) \leftrightarrow E(B))$ is derivable. An instance of (Int1) is $\Box(E(A) \leftrightarrow E(B)) \to \Box((E(A) \wedge F) \leftrightarrow (E(A) \wedge F))$. By simple transitivity moves, $\Box(A \leftrightarrow B) \to \Box((E(A) \wedge F) \leftrightarrow (E(B) \wedge F))$ is derivable.

The case where D is $E \vee F$ is similar. □

We then use the fact recorded in lemma 5 to obtain the following.

Lemma 8 *In* $B.Int$, $\Box\Big(((A \wedge B) \vee B) \leftrightarrow ((A \wedge B) \vee B)\Big) \leftrightarrow \Box(B \leftrightarrow B)$ *is derivable.*

Proof As noted in lemma 5, $((A \wedge B) \vee B) \leftrightarrow B$ is derivable. By some simple transitivity moves, it then follows from the derivability of $(B \leftrightarrow B) \leftrightarrow (B \leftrightarrow B)$ that $\Big(((A \wedge B) \vee B) \leftrightarrow ((A \wedge B) \vee B)\Big) \leftrightarrow (B \leftrightarrow B)$ is derivable. One application of (Cong) then results in the desired formula. □

The previous lemmas suffice for the following corollary.

Corollary 5 *There are violations of variable-sharing in* B.Int.

Proof *By lemma 7,* $\Box(p \leftrightarrow p) \to \Box(((p \wedge q) \vee q) \leftrightarrow ((p \wedge q) \vee q))$ *is derivable. Using simple transitivity moves with lemma 8, we obtain* $\Box(p \leftrightarrow p) \to \Box(q \leftrightarrow q)$*.* □

As with the variable-sharing violation from the (Ext1) and (Ext2) axioms, the proofs did not require much from the base logic. The modal resources required are fairly minimal as well, using only the rule (Cong). In fact, most standard relevant modal logics use the stronger rule (Mono), $A \to B \Rightarrow \Box A \to \Box B$. Therefore, avoiding hyperintensional contexts in the vocabulary $\{\wedge, \vee\}$ while still satisfying the variable-sharing criterion would require severe cuts to the logic. Given that satisfying the variable-sharing criterion is a necessary condition for being a relevant logic, a wide array of modal extensions of standard relevant logics will contain hyperintensional contexts.

With these results in hand, let us turn to some further concepts for classifying formula contexts and some discussion.

6 Discussion

In this section, we will begin by discussing some further concepts for classifying formula contexts.

Odintsov and Wansing (2021) adopt an alternative notion of hyperintensionality, using *self-extensionality*,[33] also known as *congruentiality*,[34] which they argue is closer to the suggestions of Cresswell (1975).[35] We will call the concept they use $\vdash_L$ -congruentiality, where $\vdash_L$ is the consequence relation of L.[36]

[33] See Wójcicki (1988, 342), who uses the term 'selfextensional', Font (2016, ch. 7), Avron (2017), for example. Thanks to Rohan French and Andrew Tedder for references.

[34] See Humberstone (2016, 19), among others.

[35] In this paper, we do not address the interpretive point concerning Cresswell's article.

[36] We have been discussing logics as sets of logical truths, or theorems, but discussion of Odintsov and Wansing's proposal requires using consequence relations. A single set of logical truths can be associated with multiple consequence relations, so some care must be taken in moving from L viewed as a set of logical truths to a consequence relation.

Definition 5 $\vdash_{\mathsf{L}}$*-congruentiality*

A formula context (C, p) is $\vdash_{\mathsf{L}}$*-congruential iff for all formulas A and B,*

- *if* $A \vdash_{\mathsf{L}} B$ *and* $B \vdash_{\mathsf{L}} A$*, then* $C(A) \vdash_{\mathsf{L}} C(B)$ *and* $C(B) \vdash_{\mathsf{L}} C(A)$*.*

A logic L *is* $\vdash_{\mathsf{L}}$*-congruential iff all formula contexts are* $\vdash_{\mathsf{L}}$*-congruential.*

A context is hyperintensional, in this sense, iff it is not $\vdash_{\mathsf{L}}$-congruential. The idea behind this concept is that the phenomenon of hyperintensionality is concerned with distinguishing logically equivalent formulas. We can adapt their definition to the present setting, defining $\leftrightarrow$-congruentiality, following Humberstone (2011 484–485).[37]

Definition 6 $\leftrightarrow$*-congruentiality. A formula context (C, p) is* $\leftrightarrow$*-congruential (in* L*) iff for all formulas A and B,*

- *if* $\vdash_{\mathsf{L}} A \leftrightarrow B$*, then* $\vdash_{\mathsf{L}} C(A) \leftrightarrow C(B)$*.*

A logic L is $\leftrightarrow$-congruential iff all formula contexts are congruential in L.

Relevant logics and their usual modal extensions are $\leftrightarrow$-congruential, although there are modal extensions which are not.[38] We can make this claim more precise.

Theorem 3 *Let* L *be any sublogic of* R *extending* B *that is closed under the rules of* B*. Let* L.Cong *be* L *extended by the rule* (Cong)*. Then every formula context in the vocabulary* $\{\neg, \rightarrow, \wedge, \vee, \Box\}$ *is* $\leftrightarrow$*-congruential in* L.Cong.

Proof The proof is by induction on the complexity of the context C. The base cases are immediate, and the cases for the non-modal connectives are handled by the rules and axioms of B. The case where C has the form $\Box D$ is handled by the rule (Cong). □

In particular, the context $(p \wedge r, p)$ is $\leftrightarrow$-congruential while also being hyperintensional.

While Odintsov and Wansing think that failure of $\vdash_{\mathsf{L}}$-congruentiality is the proper concept for formalizing the phenomenon of hyperintensionality, it is worth distinguishing failures of $\vdash_{\mathsf{L}}$-congruentiality, or failures of $\leftrightarrow$-congruentiality, and hyperintensionality, as defined in §3, for two reasons. First, hyperintensionality builds in an explicit modal element that is absent in both definitions of congruentiality in the sense that the former, but not the latter requires a modal operator be used in its defini-

[37] This adaptation is not uncommon, for which see, e.g., (Williamson, 2006, 313), but it forfeits a potential virtue of $\vdash_{\mathsf{L}}$-congruentiality, namely being free of displayed connectives and instead involving only the salient consequence relation, rather than also involving various connectives. There is no apparent way to adapt the definition of hyperintensionality, in the sense from §3, to be free of connectives, as it is not clear how one would maintain the modal element appropriately.

[38] See Savić and Studer (2019) and Standefer (2023c) for examples.

tion. It is not clear how one would add a modal element to $\vdash_{\mathsf{L}}$-congruentiality, and there are apparently different options for inserting modalities into $\leftrightarrow$-congruentiality.

The second reason is that once one is working with a modal logic for which the rule (Nec), $A \Rightarrow \Box A$, fails, the relationship between hyperintensionality and failures of congruentiality becomes more complex, as noted by Williamson (2006). This is particularly salient in the present context, because (Nec) fails in many relevant modal logics, and it is not required by the relevant analogs of Kripke models.[39]

The relationship between hyperintensionality and failures of congruentiality becomes more complex with certain extensions of the language as well. With certain extensions of the language, some contexts can fail to be congruential but also be non-hyperintensional. As noted by Williamson (2006, 315), the addition of an actuality operator $\mathbb{A}$ to the language can render $(\Box p, p)$ a non-congruential yet non-hyperintensional context.[40] This example suggests that as the language is enriched, the relationship between non-congruentiality and hyperintensionality will become more complex, as they track slightly different features of the logics.

Presenting congruentiality and hyperintensionality as rivals is, however, artificial. One can use both concepts for classifying formula contexts, and using one does not preclude using the other. They are both interesting and important. It is, we think, worth distinguishing them, and they could potentially be put to different logical uses.

It is worth pointing out a feature of relevant logics that is, in some ways, similar in spirit to hyperintensionality. Classical logic is *monothetic* in the sense that for any two logical truths A and B, $A \leftrightarrow B$ is a logical truth.[41] From the point of view of classical logic, there is only a single logical truth. HYPE is also monothetic, replacing the classical biconditional with the biconditional of HYPE, and similarly for intuitionistic logic. Relevant logics are *polythetic* meaning that there are non-equivalent logical truths, that is, there are logical truths A and B such that $A \leftrightarrow B$ is not a logical truth.[42] As an example, we note that $\vdash_{\mathsf{R}} p \to p$ and $\vdash_{\mathsf{R}} q \to q$, but $R(p \to p) \leftrightarrow (q \to q)$. In this sense, relevant logics permit one to draw distinctions between logical truths. By contrast, any logic that obeys the weakening rule, $A \Rightarrow B \to A$, will be monothetic.

There is a special case of being polythetic that is worth bringing out. Let us say that a formula A is a *classical tautology* in the vocabulary $\{\neg, \wedge, \vee\}$ iff there is a formula C whose connectives are all from the set $\{\neg, \wedge, \vee\}$ such that C is a classical tautology and A can be obtained from C by substituting formulas for atoms. Consider two formulas A and B that are classical tautologies in the vocabulary $\{\neg, \wedge, \vee\}$. In R, there are classical tautologies that are theorems but not equivalent. For example, $\vdash_{\mathsf{R}} p \vee \neg p$ and $\vdash_{\mathsf{R}} q \vee \neg q$, but those two instances of excluded middle are not equiva-

39 See Fuhrmann (1990) or Standefer (2025b) for more details.

40 To elaborate, Crossley and Humberstone (1977) distinguish two kinds of validity when actuality is in the language, real world validity and general. One kind, real world validity, renders $\mathbb{A}B \leftrightarrow B$ valid. Despite this, $\Box\mathbb{A}q \leftrightarrow \Box q$ is not valid, whence $(\Box p, p)$ is non-congruential. Verifying non-hyperintensionality will be left to the reader.

For discussion of real world and general validity, see Zalta (1988), Hanson (2006), Zalta and Nelson (2012), and French (2012), among others.

41 See Humberstone (2011, 231).

42 This point was also made by Standefer (2019), albeit in a discussion of justification logics.

lent, as $R(p \vee \neg p) \leftrightarrow (q \vee \neg q)$. This last biconditional must fail, on pain of violating variable-sharing. Thus even in the relevant logics in which all classical tautologies are theorems, such as R, relevant logics can draw distinctions among those tautologies.[43] This marks an important difference with, say, HYPE. In HYPE one can distinguish classical tautologies, such as $p \vee \neg p$ and $q \vee \neg q$, since they are not theorems of HYPE. Any two formulas that are theorems of HYPE are also equivalent in HYPE. In contrast, relevant logics can draw distinctions among classical tautologies, even when those tautologies are theorems.

Drawing distinctions among valid formulas and classical tautologies carries over to modal extensions of the relevant logics as well. The logics can draw distinctions among classical modal logical truths, including those that have been necessitated. One can have an S5-type extension of R that still distinguishes $\Box(p \vee \neg p)$ and $\Box(q \vee \neg q)$ while having both as theorems. Just as different *logical truths* may not imply each other, different *necessary* logical truths may not imply each other either. While this feature is different from hyperintensionality and (failures of) congruentiality, it formalizes a similar idea, namely that of drawing distinctions among necessary or logical truths.

The results of this paper show that almost all the common modal extensions of relevant logics have hyperintensional contexts. This result extends to HYPE, although the range of such contexts appears more limited there than for R. As one weakens the logic, the range of hyperintensional contexts grows, a feature that extends to HYPE and other substructural logics as well. Hyperintensionality is of interest in a wide range of philosophical applications of logic, such as logics of belief and epistemic logics. In epistemic logics, we think of '$\Box$' as representing knowledge, so that '$\Box p$ ' should be understood as saying 'the agent knows p' and in doxastic logics '$\Box p$' should be understood as saying 'the agent believes p'.[44] The most common modal principles for doxastic and epistemic logics are consequences of the TRIV principles. This means that adding the standard principles for doxastic and epistemic logics to, say, R will result in a sublogic of R.TRIV. This, in turn, means that one of the hyperintensionality result will straightforwardly apply to these logics.

In the context of epistemic logic, a context being hyperintensional means that an agent's knowledge will not be closed under known equivalence in the following sense. The agent knowing that $p \leftrightarrow q$ need not imply that they know $(p \wedge r) \leftrightarrow (q \wedge r)$, to use the example from theorem 1. If one adopts a base logic weaker than R, one will have more examples where knowing one equivalence does not lead to knowing others, such as knowing $p \leftrightarrow q$ but not knowing $\neg p \leftrightarrow \neg q$.[45]

Such failures open up possibilities for representing agents that are limited in different ways in how they can use their knowledge. It is natural, in this context, to represent an agent's knowledge as a logical theory, which is a set of formulas closed

[43] There are many standard relevant logics in which classical tautologies in the vocabulary $\{\neg, \wedge, \vee\}$ are not theorems. B is an example of such, but one does not have to weaken the logic that much for examples. See Slaney (1984) for discussion.

[44] See Meyer and van der Hoek (1995), van Ditmarsch et al. (2015), or Rendsvig et al. (2023) for an introduction to epistemic logic.

[45] Reasons why some of these equivalences fail might be obtained from recent work on topics and topic-transforming operators, such as that of Berto (2022), Ferguson and Logan (2025), and Tedder (2025).

under provable implications and adjunction. More carefully, a theory in the logic L is a set X of formulas such that (i) if $\vdash_{\mathsf{L}} A \to B$ and $A \in X$, then $B \in X$ and (ii) if $A \in X$ and $B \in X$, then $A \wedge B \in X$. A context being hyperintensional means that instance of the hyperintensionality scheme is not provable. This in turn gives rise to theories containing the antecedent but not the consequent, representing agents whose knowledge is not closed under substitutions of certain known equivalences.[46] Additionally, there will be theories in which the agent's knowledge will be closed under some equivalences, but not others. As an example, there will be a theory in which the agent knows $p \leftrightarrow q$ but not $(p \wedge r) \leftrightarrow (q \wedge r)$ and the agent knows both $p \leftrightarrow s$ and $(p \wedge r) \leftrightarrow (s \wedge r)$, which is to say $\Box(p \leftrightarrow q)$, $\Box(p \leftrightarrow s)$, and $\Box((p \wedge r) \leftrightarrow (s \wedge r))$ are all in the theory but $\Box((p \wedge r) \leftrightarrow (q \wedge r))$ is not. While an agent's knowledge will still be closed under provable implications and equivalences, there will be a lot of flexibility in representing an agent's knowledge of true, or merely assumed true, equivalences.

One direction for future work that would be useful in the development of the epistemic logic application suggested above is precisely characterizing the range of hyperintensional contexts in the different relevant and substructural logics. This would be useful in better understanding the ways in which non-classical epistemic logics avoid, or fail to avoid, problems of logical omniscience.[47]

The hyperintensionality discussed above is the sort that arises naturally in the logics under consideration. It arises in an axiomatic setting from adding a selection of more or less standard axioms to a base relevant logic. It arises naturally in the context of models as well, if one uses the straightforward approach of adding a binary modal accessibility relation to a model for a relevant logic. This is all to say that the hyperintensionality described above arises without the need for any additional logical machinery or any formal "funny business." Making the bog standard extensions of relevant logics, or indeed a range of substructural logics, with modal operators will result in hyperintensional contexts. One can, of course, appeal to various modeling techniques used to obtain hyperintensional contexts over classical logic to obtain such contexts in relevant logics.[48] These modeling techniques will likely interact with the natural hyperintensionality of relevant logics in interesting ways.

To summarize, relevant logics are hyperintensional, in at least one important sense, when considering many natural extensions with necessity operators. Related to the hyperintensional contexts, there are failures of extensionality for relevant logics, and the range of non-extensionality and hyperintensionality grows as one weakens the logic. Relevant logics and their modal extensions are, generally, congruential, so they are not hyperintensional in the sense preferred by Odintsov and Wansing. Nonetheless, we do agree with Odintsov and Wansing's closing suggestion to study non-self-

[46] Given the results concerning $\leftrightarrow$-congruentiality at the start of this section, we want to emphasize these are not provable equivalences.

[47] See, for example, Sedlár (2015, 2016), Standefer et al., (2023), and Ferenz (2023), among others, for some discussion of logical omniscience in non-classical settings. For a contrasting recent discussion of omniscience in the setting of classical logic, see Hawke, Özgün, and Berto (2020).

[48] See Sedlár (2019) for a general framework for hyperintensionality in the context of classical logic. Application and generalization of this framework to relevant logics has, as far as we know, not been explored.

extensional, or non-congruential, operators, as non-classical logics likely have much to contribute areas in which they are used. Despite being congruential, relevant logics are polythetic, which allows them to draw distinctions among logical truths in ways reminiscent of hyperintensionality. Finally, we suggest that it would be worth exploring the use of this hypreintensionality in the context of epistemic and doxastic logics, as there is general interest in the phenomenon of hyperintensionality in those areas.

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