Our Aim

To introduce *proof theory*, with a focus in its applications in philosophy, linguistics and computer science.
Our Aim for Today

Examine the connections between proof theory and semantics, both formal model theory, and more general philosophical considerations concerning meaning.
Today's Plan

Speech Acts and Norms

Proofs and Models

Beyond
SPEECH ACTS AND NORMS
An idea found in Brandom’s *Making It Explicit* is that the meaning of linguistic items should first be understood in terms of their use.

The linguistic (conceptual) practices of communities set up norms governing their behavior.

These practices have features that we can make explicit through the introduction of new vocabulary.
The rules that govern a connective are taken to *define* the new connective.

This appears to make it really easy to introduce new logical terms.

Specify a set of rules governing a connective, and you’ve got a new connective.
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Specify a set of rules governing a connective, and you’ve got a new connective.

But, there’s a problem.
Arthur Prior pointed out that if a set of rules is enough to define a connective, then *tonk* is legitimate.
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\[
\begin{align*}
X, A \rightarrow C & \quad [\text{tonkL}] \\
\frac{X, A \otimes B \rightarrow C}{X, A \otimes B \rightarrow C} & \\
\frac{X \rightarrow B}{X \rightarrow A \otimes B} & \quad [\text{tonkR}] \\
\frac{B \rightarrow B}{B \rightarrow A \otimes B} & \quad [\text{tonkR}] \\
\frac{A \rightarrow A}{A \otimes B \rightarrow A} & \quad [\text{tonkL}] \\
\frac{A \otimes B \rightarrow A}{B \rightarrow A} & \quad [\text{Cut}]
\end{align*}
\]
Responding to tonk

Nuel Belnap responded to Prior’s article, saying that additional conditions need to be satisfied in order to define a connective.

Connectives aren’t introduced out of thin air, there is a context of deducibility, e.g. the full set of Gentzen’s structural rules.

In order to be a definition, an extension has to be conservative, while tonk manifestly is not.

In order to be a definition, an addition has to be uniquely specified.

These ideas have been taken up and developed by Dummett and others in discussions of harmony.
Defining rules

Running with Belnap’s idea, we can define connectives using the following double line rules

\[
\frac{X \succ A, Y}{X, \neg A \succ Y} \quad \text{[\neg Df]}
\]

\[
\frac{X, A \succ B, Y}{X \succ A \rightarrow B, Y} \quad \text{[\rightarrow Df]}
\]

\[
\frac{X \succ A, B, Y}{X \succ A \lor B, Y} \quad \text{[\lor Df]}
\]

\[
\frac{X \succ A|_n, Y}{X \succ (\forall x)A, Y} \quad \text{[\forall Df]}
\]

\[
\frac{X, A|_n \succ Y}{X, (\exists x)A \succ Y} \quad \text{[\exists Df]}
\]

\[
\frac{X, Fs \succ Ft, Y}{X \succ s = t, Y} \quad \text{[=} Df]
\]

(Provided \(n\) and \(F\) are not present in \(X\) and \(Y\))
Defining rules

Concepts defined using defining rules have the following features

- They’re uniquely defined. Two concepts defined using the same rule are interderivable, given Identity and Cut.
- They can be used to generate the other introduction rules using Identity and Cut.
- They’re conservatively extending.
Uniqueness

Suppose $\land$ and $\&$ both obey the defining rule for $\land$

\[
\begin{align*}
\text{[Id]} & \quad A \land B \vdash A \land B \\
\text{[^Df]} & \quad A, B \vdash A \land B \\
& \quad A \land B \vdash A \land B \\
\text{[&Df]} & \quad A \land B \vdash A \land B \\
\text{[\land Df]} & \quad A \land B \vdash A \land B
\end{align*}
\]
Suppose we want to get the right conjunction rule

\[ X \vdash A, Y \quad X \vdash B, Y \]
\[ \frac{}{X \vdash A \land B, Y}^{\land R} \]

We proceed as follows

\[ X \vdash B, Y \]
\[ \frac{A \land B \vdash A \land B}{\frac{X \vdash B, Y \quad A, B \vdash A \land B}{\frac{X \vdash A, Y \quad X, A \vdash A \land B, Y}{\frac{}{X \vdash A \land B, Y}^{\land Df} \quad \frac{}{X, A \vdash A \land B, Y}^{\land Df} \quad \frac{}{X \vdash A \land B, Y}^{\land Df}}^{\land Df}}^{\land Df} \]

\[ \frac{}{X \vdash A \land B, Y}^{\land Df} \]
Many philosophers and logicians take *assertion* to be the primary speech act, which is used to define others.

Others argue that *denial* should be understood as a primitive act on its own.

We take logic, in particular valid sequents, as presenting normative relations between assertions and denials.

\[ X \supset Y \] tells us that one should not assert everything in \( X \) while denying everything in \( Y \).
Positions

\[ X \succ Y \]
Positions

X \not\rightarrow Y
Positions

Invalid sequents can be viewed as positions in a discourse
What do the structural rules say in terms of assertion and denial?
\( A \rightarrow \neg A \)

Asserting \( A \) clashes with denying \( A \)
If asserting $X, Y$ clashes with denying $Z$, then asserting more stuff still clashes.
Structural Rules

\[
\begin{align*}
X, A, AY & \not\triangleright Z \quad [WL] \\
\hline
X, A, Y & \not\triangleright Z \\
X \not\triangleright Y, A, A, Z & \quad [WR] \\
\hline
X \not\triangleright Y, A, Z
\end{align*}
\]

If asserting or denying \( A \) twice results in a clash, then asserting or denying \( A \) just once results in a clash.
Structural Rules

\[
\begin{align*}
X, A, B, Y & \succ Z & \text{[CL]} \\
X, B, A, Y & \succ Z
\end{align*}
\]

\[
\begin{align*}
X & \succ Y, A, B, Z & \text{[CR]} \\
X & \succ Y, B, A, Z
\end{align*}
\]

If some assertions and denials clash, then asserting and denying the same things in a different order still clashes.
Structural Rules

\[
\begin{align*}
X &\not\rightarrow Y, A & A, X &\not\rightarrow Y \\
\hline
\end{align*}
\]

[Cut]

\[X \not\rightarrow Y\]

If asserting \(X\) and denying \(A\) and \(Y\) clashes, and asserting \(X\) and \(A\) while denying \(Y\) clashes, then asserting \(X\) and denying \(Y\)

Contrapositively, if asserting \(X\) and denying \(Y\) does not clash, then either asserting \(X\) and \(A\) while denying \(Y\) does not clash or asserting \(X\) while denying \(Y\) and \(A\) does not clash
Belnap argued that a systematic logical treatment of language should give equal weight to imperatives and interrogatives.

1. THE DECLARATIVE FALLACY

My thesis is simple: systematic theorists should not only stop neglecting interrogatives and imperatives, but should begin to give them equal weight with declaratives. A study of the grammar, semantics, and pragmatics of all three types of sentence is needed for every single serious program in philosophy that involves giving important attention to language.

Attempting to understand all linguistic behavior in terms of assertions commits the Declarative Fallacy.

The hope is that the view of sequents and logic can be extended to other speech acts.
PROOFS AND MODELS
How might *truth* enter this picture?

Models are ways of systematically elaborating finite positions into ideal, infinite positions that settle every proposition.

In the propositional case, valuations are generated by ideal positions.
Positions to models

The members of $X$ are true and the members of $Y$ are false
Positions to models

The members of $X$ are true and the members of $Y$ are false (relative to $[X : Y]$).
Example

\[ [p \lor q, \ r : \neg p] \]
Example

\[ [p \lor q, r : \neg p] \]

\[ p \lor q, r \]

\[ p \lor q, r \]
Example

\[ [p \lor q, r : \neg p] \]

\[
\begin{align*}
  & \quad p \lor q, r \quad true \\
\end{align*}
\]
Example

\[ [p \lor q, r : \neg p] \]

\[ \begin{array}{c}
\neg p \\
\hline
p \lor q, r \quad true \\
\hline
\end{array} \]
Example

\[ [p \lor q, \ r : \neg p] \]

\[ \begin{align*}
  &p \lor q, \ r & \text{true} \\
  &\neg p & \text{false}
\end{align*} \]
Example

\[ [p \lor q, r : \neg p] \]

\[ \begin{align*}
  & p \lor q, r \quad true \\
  & \neg p \quad false \\
  & p
\end{align*} \]
Example

\[ [p \lor q, r : \neg p] \]

\[
\begin{array}{c|c|c}
  p \lor q, r & true \\
  \neg p & false \\
  p & ??? \\
\end{array}
\]
Example

\[[p \lor q, \ r : \neg p]\]

\[\begin{align*}
  p \lor q, \ r & \quad \text{true} \\
  \neg p & \quad \text{false} \\
  p & \quad \text{??}
\end{align*}\]

**Definition:** \(A\) is *true* at \([X : Y]\) iff \(X \succ A, Y\).

**Definition:** \(A\) is *false* at \([X : Y]\) iff \(X, A \succ Y\).
Example

\[ [p \lor q, r : \neg p] \]

\[ \begin{array}{l}
 p \lor q, r \quad true \\
 \neg p \quad false \\
 p \quad true
\end{array} \]

**Definition:** A is *true* at \([X : Y]\) iff \(X \rhd A, Y\).

**Definition:** A is *false* at \([X : Y]\) iff \(X, A \rhd Y\).
Example

\[
[p \lor q, r : \neg p]
\]

\[
\begin{array}{c}
p \lor q, r & \text{true} \\
\neg p & \text{false} \\
p & \text{true} \\
p \land r \\
\end{array}
\]

**Definition:** \( A \) is *true* at \([X : Y]\) iff \( X \vdash A, Y \).

**Definition:** \( A \) is *false* at \([X : Y]\) iff \( X, A \not\vdash Y \).
Example

\[ [p \lor q, r : \neg p] \]

\[
\begin{array}{c|c}
\hline
\text{p } \lor \text{q, r} & \text{true} \\
\hline
\neg p & \text{false} \\
\hline
p & \text{true} \\
\hline
p \land r & \text{true} \\
\hline
\end{array}
\]

**Definition:** A is *true* at \([X : Y]\) iff \(X \supset A, Y\).

**Definition:** A is *false* at \([X : Y]\) iff \(X, A \supset Y\).
A \land B \text{ is true at } [X : Y] \text{ iff } A \text{ and } B \text{ are true at } [X : Y].$

A \lor B \text{ is false at } [X : Y] \text{ iff } A \text{ and } B \text{ are false at } [X : Y].

\neg A \text{ is true at } [X : Y] \text{ iff } A \text{ is false at } [X : Y].

\neg A \text{ is false at } [X : Y] \text{ iff } A \text{ is true at } [X : Y].
Classical Logic

\( A \land B \) is true at \([X : Y]\) iff \( A \) and \( B \) are true at \([X : Y]\).

\( A \lor B \) is false at \([X : Y]\) iff \( A \) and \( B \) are false at \([X : Y]\).

\( \neg A \) is true at \([X : Y]\) iff \( A \) is false at \([X : Y]\).

\( \neg A \) is false at \([X : Y]\) iff \( A \) is true at \([X : Y]\).

However, \( p \land q \) is false at \([p \land q]\)
Classical Logic

\( A \land B \text{ is true at } [X : Y] \text{ iff } A \text{ and } B \text{ are true at } [X : Y]. \)

\( A \lor B \text{ is false at } [X : Y] \text{ iff } A \text{ and } B \text{ are false at } [X : Y]. \)

\( \neg A \text{ is true at } [X : Y] \text{ iff } A \text{ is false at } [X : Y]. \)

\( \neg A \text{ is false at } [X : Y] \text{ iff } A \text{ is true at } [X : Y]. \)

However, \( p \land q \text{ is false at } [ : p \land q] \) but neither \( p \) nor \( q \) is false at \([ : p \land q]\) since neither \( p \rhd p \land q \) nor \( q \rhd p \land q. \)
Classical Logic

\[ A \land B \text{ is true at } [X : Y] \iff A \text{ and } B \text{ are true at } [X : Y]. \]
\[ A \lor B \text{ is false at } [X : Y] \iff A \text{ and } B \text{ are false at } [X : Y]. \]
\[ \lnot A \text{ is true at } [X : Y] \iff A \text{ is false at } [X : Y]. \]
\[ \lnot A \text{ is false at } [X : Y] \iff A \text{ is true at } [X : Y]. \]

However, \( p \land q \) is false at \([ : p \land q]\) but neither \( p \) nor \( q \) is false at \([ : p \land q]\) since neither \( p \rhd p \land q \) nor \( q \rhd p \land q \).

Similarly, \( r \) is neither true nor false at \([p : q]\).
FACT: If $A$ is neither true nor false in $[X : Y]$ then both $[X, A : Y]$ and $[X : A, Y]$ is invalid, and each position settles $A$ — one as true and the other as false.
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So, if $[X : Y]$ doesn’t settle the truth of a statement $A$, then we can throw $A$ in on either side, to get a more comprehensive position which *does* settle it.
Extensions

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In general, if $X \not\rightarrow Y$ then either $X, A \not\rightarrow Y$ or $X \not\rightarrow A, Y$. 
FACT: If $A$ is neither true nor false in $[X : Y]$ then both $[X, A : Y]$ and $[X : A, Y]$ is invalid, and each position settles $A$ — one as *true* and the other as *false*.

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In general, if $X \not\vdash Y$ then either $X, A \not\vdash Y$ or $X \not\vdash A, Y$.

$$
\begin{align*}
X \not\vdash Y, A & \quad A, X \not\vdash Y \\
\hline
X \not\vdash Y & \quad \text{[Cut]}
\end{align*}
$$
Maximal Positions

A maximal position is the limit of the process of throwing in each sentence in either the left or the right hand side. You can think of it as:

A pair \([X:Y]\) of infinite sets, such that \(X \not\subset Y\) and \(X \cup Y\) is the whole language.

**Fact:** Every maximal position makes each sentence either true or false.

**Fact:** If \(X \not\subset Y\), there's a maximal \([X:Y]\) extending \([X:Y]\).
Maximal Positions

\([X : Y]\) is finitary, where \(X\) and \(Y\) are sets (or multisets or lists ...).
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▶ A pair [X : Y] of infinite sets, such that X ∉ Y and X ∪ Y is the whole language.

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A maximal position is the limit of the process of throwing in each sentence in either the left or the right hand side. You can think of it as:

- A pair \([X : Y]\) of infinite sets, such that \(X \not\subseteq Y\) and \(X \cup Y\) is the whole language.

**Fact**: Every maximal position makes each sentence either *true* or *false*.

**Fact**: If \(X \not\subseteq Y\), there’s a maximal \([X' : Y]\) extending \([X : Y]\).
Assign truth values relative to *maximal* positions.
Assign truth values relative to *maximal* positions.

In a slogan, *truth value = location in a maximal position,*
The ideal position construction handles classical logic

With some small adjustments, it can be used to provide models for intuitionistic logic

The system LJ is single-conclusion, but there is an intuitionistic sequent system that has multiple conclusions

The construction with these two systems yield Kripke models and Beth models
The hypersequent system for S5 can be used to give a similar construction.

Each component of a hypersequent describes a possible world.
S5 hypersequents

\[
\frac{\mathcal{H}[X \triangleright Y \mid X', A \triangleright Y']}{\mathcal{H}[X, \Box A \triangleright Y \mid X' \triangleright Y']} \quad \text{[\(\Box L\)]}
\]

\[
\frac{\mathcal{H}[X \triangleright Y \mid A \triangleright \ ]}{\mathcal{H}[\Diamond A, X \triangleright Y']} \quad \text{[\(\Diamond L\)]}
\]

\[
\frac{\mathcal{H}[X \triangleright Y \mid \triangleright A]}{\mathcal{H}[X \triangleright \Box A, Y]} \quad \text{[\(\Box R\)]}
\]

\[
\frac{\mathcal{H}[X \triangleright Y \mid X' \triangleright A, Y']}{\mathcal{H}[\Diamond A, X \triangleright Y \mid X' \triangleright Y']} \quad \text{[\(\Diamond R\)]}
\]
Extending positions

Invalid sequents \([X : Y]\)

Invalid hypersequents \([[X : Y], [X' : Y'], \ldots]]\)

Say one set of pairs \(H\) extends another \(G\), \(G \preceq H\), just in case for each component \([X : Y]\) in \(G\), there is a component \([U : V]\) in \(H\) such that \(X \leq U\) and \(Y \leq V\).

Example: \(f([p : q]; [s : r])\) is extended by both \(f([p, s : r, q, t])\) and \(f([p, t : q]; [s : r, p])\).
Invalid sequents \([X : Y]\)

Invalid hypersequents \([[[X : Y], [X' : Y']], \ldots]\)

Say one set of pairs \(\mathcal{H}\) extends another \(\mathcal{G}\), \(\mathcal{G} \preceq \mathcal{H}\), just in case for each component \([X : Y]\) in \(\mathcal{G}\), there is a component \([U : V]\) in \(\mathcal{H}\) such that \(X \subseteq U\) and \(Y \subseteq V\)

Example: \(\{[p : q], [s : r]\}\) is extended by both \(\{[p, s : r, q, t]\}\) and by \(\{[p, t : q], [s : r, p]\}\)
Where are the truth values now?
Where are the truth values now?

Maximal positions \([\mathcal{X} : \mathcal{Y}]\)

Maximal modal positions \([[[\mathcal{X} : \mathcal{Y}], [\mathcal{X}' : \mathcal{Y}']], \ldots]\)
Where are the truth values now?

Maximal positions \([\mathcal{X} : \mathcal{Y}]\)

Maximal modal positions\([\mathcal{X} : \mathcal{Y}], [\mathcal{X}' : \mathcal{Y'}], \ldots]\)

A set of pairs \(\mathcal{H}\) is a modal position iff there is no valid hypersequent
\[X_1 \succ Y_1 \mid \cdots \mid X_n \succ Y_n\] extended by \(\mathcal{H}\)

A modal position \(\mathcal{H}\) is maximal iff there is no modal position \(\mathcal{I}\) such that \(\mathcal{H} \prec \mathcal{I}\)
Building maximal modal positions

The process of expanding a modal position can add formulas to a component as well as adding more components.

Some maximal modal positions, however, will contain finitely many components.

The construction builds connected chunks of the $S5$ canonical model, taking the accessibility relation to be an equivalence relation rather than the universal relation.
As in the classical case, the Cut rule adds new formulas to individual components.

The modal rules can extend a position with new components.
Building maximal modal positions

\[
\begin{align*}
\mathcal{H}[X \rightsquigarrow Y \mid X', A \rightsquigarrow Y'] & \quad [\Box L] \\
\mathcal{H}[X, \Box A \rightsquigarrow Y \mid X' \rightsquigarrow Y'] & \\
\mathcal{H}[X \rightsquigarrow Y \mid \Box A] & \quad [\Box R] \\
\mathcal{H}[X \rightsquigarrow \Box A, Y] & 
\end{align*}
\]

If \([X : \Box A, Y], [X_i : Y_i]\) isn’t derivable, then \([X : \Box A, Y], [\vdash A], [X_i : Y_i]\) can’t be either.

If the latter were derivable then the former would be by \([\Box R]\)

Similarly but for \([\Box L]\), if, e.g. \([X, \Box A : Y], [X' : Y'], [X_i : Y_i]\) isn’t derivable, then \([X, \Box A : Y], [X', A : Y'], [X_i : Y_i]\) can’t be.
Necessity in maximal modal positions

For a maximal modal position \{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\},
\[\Box A\] is true at \([\mathcal{X}_i : \mathcal{Y}_i]\) iff \(A\) is true at each \([\mathcal{X}_j : \mathcal{Y}_j], j \in I\)

(\Rightarrow) If \(\Box A\) is true at \([\mathcal{X}_i : \mathcal{Y}_i]\) and \(A\) were not true at some component \([\mathcal{X} : \mathcal{Y}]\), then since \([\mathcal{X} : \mathcal{Y}]\) is a maximal position, we would have \(A \in \mathcal{Y}\) but \(\Box A \vdash A\) is a valid sequent (by \([\Box L]\) from the axiom \(\vdash \Box A \vdash A\)), so \([\mathcal{X}_i : \mathcal{Y}_i], [\mathcal{X}_j : \mathcal{Y}_j]\) would not be a position, as \(\Box A \in \mathcal{X}_i\) and \(A \in \mathcal{Y}_j\), so \([\mathcal{X}_i : \mathcal{Y}_i] : i \in I\) isn’t a position. As it is, whenever \(\Box A \in [\mathcal{X}_i : \mathcal{Y}_i]\), \(A\) is true at every component.
Necessity in maximal modal positions

For a maximal modal position \{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\},
\[ \square A \text{ is true at } [\mathcal{X}_i : \mathcal{Y}_i] \iff A \text{ is true at each } [\mathcal{X}_j : \mathcal{Y}_j], j \in I \]

(\Leftarrow) Suppose \[\square A \text{ isn't true at } [\mathcal{X}_i : \mathcal{Y}_i]. \] So we have \[\square A \in \mathcal{Y}_i. \] Take \[\{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\} \cup [ : A]. \] This is a position. Suppose that it is not. Then there is a derivable hypersequent \(\Theta \vdash [\mathcal{X} \Rightarrow Y] H,\) where \(X \subseteq \mathcal{X}_i, Y \subseteq \mathcal{Y}_i\) and \(H\) is extended by the other components of the modal position. If that were the case, then by \([\square R],\) we could derive \(X \rightarrow \square A, Y \mid H,\) but that is extended by the original modal position. It is, then, not valid. So, \([\mathcal{X}_i : \mathcal{Y}_i] : i \in I\} \cup [ : A] \text{ is a position, so it is extended by a maximal modal position, which must be }\}
\[\{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\}, \text{ as that is not extended by any modal positions. Therefore, for some } j \in I, A \in \mathcal{Y}_j.\]
A modal position extended by a finite, maximal modal position

\[ \neg \diamond p \lor \diamond \neg p \text{ is not valid} \]

So \([ : \diamond p \lor \diamond \neg p]\) is a position

Using the rules, one obtains

\[
[ [ : ], [ : ], [ : \diamond p \lor \diamond \neg p ]] 
\]

One can then choose extensions in such a way that no additional components are needed
A modal position extended by a finite, maximal modal position

\[ \neg \Box p \lor \Box \neg p \text{ is not valid} \]

So \[ [ [ : ] , [ : ] , [ : \Box p , \Box \neg p , \Box p \lor \Box \neg p ] ] \]

Using the rules, one obtains

One can then choose extensions in such a way that no additional components are needed
A modal position extended by a finite, maximal modal position

\[ \neg \Box p \lor \Box \neg p \text{ is not valid} \]

So \[ : \Box p \lor \Box \neg p \] is a position

Using the rules, one obtains

\[ [[ : p], [ : \neg p], [ : \Box p, \Box \neg p, \Box p \lor \Box \neg p]] \]

One can then choose extensions in such a way that no additional components are needed
A modal position extended by a finite, maximal modal position

\[ \neg \Box p \lor \Box \neg p \text{ is not valid} \]

So \( [ : \Box p \lor \Box \neg p ] \) is a position

Using the rules, one obtains

\[ [[ : p], [p : \neg p], [ : \Box p, \Box \neg p, \Box p \lor \Box \neg p]] \]

One can then choose extensions in such a way that no additional components are needed
Maximality Facts

Each modal position can be extended to a maximal modal position.

Each component of a maximal modal position is a maximal position.

Each maximal modal position corresponds to a simple Kripke model: $\square A$ is true at $\{[x_i; y_i] : i \in I\}$ iff $A$ is true in every position in the modal position.
BEYOND
Further directions

There are many directions one could go from here

One could add other *connectives* and *predicates*

One could add axioms to obtain *theories*
Truth

These rules are inconsistent in classical logic, so one will need to go non-classical to hang onto them.

They take complex formulas to atomic formulas, which leads to complications for showing that Cut can be eliminated.
Arithmetic

Take a language with $=, 0, ', +, \times$

$\vdash x + 0 = x$

$\vdash x + y' = (x + y)'$

$\vdash x \times 0 = 0$

$\vdash x \times y' = (x \times y) + x$

$\vdash x' = y' \Rightarrow x = y$

$0 = x' \Rightarrow$

$\frac{X \vdash A(0), Y}{X, A(x) \vdash A(x'), Y}$

$\frac{X \vdash A(x), Y}{X, A(x') \vdash A(x), Y}$

$\frac{A(0), X \vdash Y}{A(x), X \vdash Y}$
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