5 Inferentialism, Structure, and Conservativeness

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1. Introduction

Logical inferentialism is, roughly, the view that the meaning of a logical connective is determined by the inference rules governing that connective. For example, in systems of natural deduction the introduction and elimination rules for a connective fix its meaning. The paradigmatic example is conjunction, governed by the following rules.

$$\frac{A \quad B}{A \wedge B}\ (\wedge I) \qquad \frac{A \wedge B}{A}\ (\wedge E) \qquad \frac{A \wedge B}{B}\ (\wedge E)$$

The conceptual content of conjunction is given, not by a truth-condition, but by the inferential role specified by the inference rules governing it. The introduction rule tells us under what conditions we are entitled to assert a conjunction, namely when we are entitled to assert its conjuncts, and the elimination rules tell us what we commit ourselves to when we assert a conjunction, namely, to both conjuncts.

However, it is well known that some combinations of rules for a connective can result in disaster. The most famous example is Prior's (1961) tonk, $\rhd\!\lhd$:[1]

$$\frac{A}{A \rhd\!\lhd B}\ (\rhd\!\lhd I) \qquad \frac{A \rhd\!\lhd B}{B}\ (\rhd\!\lhd E)$$

The inference rules for tonk lead to a trivial consequence relation where any conclusion is derivable from any (non-empty) set of premises. So if logical inferentialism is correct, there has to be some constraint on which inference rules can confer meaning on a connective. In an early response to Prior, Belnap (1962) suggested a constraint that has proved influential: the inferentialist should require that the addition of logical vocabulary to a consequence relation should be a conservative extension. An extension L' of a theory L is *conservative* when the language of L' contains that of L and $X \Rightarrow_{L'} A$ only if $X \Rightarrow_L A$, when X and A are in the language of L. The condition will rule out tonk as a legitimate expression, assuming that the original theory has a nontrivial standard consequence relation.

Robert Brandom, following the work of Michael Dummett, endorses conservativeness as a requirement. Among the reasons Brandom gives for requiring that extensions be conservative is the expressive role of logic:

> But the expressive account of what distinguishes logical vocabulary shows us a deep reason for this demand [of conservative extension]; it is needed not only to avoid horrible consequences but also because otherwise logical vocabulary cannot perform its expressive function. Unless the introduction and elimination rules are inferentially conservative, the introduction of the new vocabulary licenses new material inferences, inferences good in virtue of the concepts involved rather than their form, and so alters the contents associated with the old vocabulary. So if logical vocabulary is to play its distinctive expressive role of making explicit the original material inferences, and so conceptual contents expressed by the old vocabulary, it must be a criterion of adequacy for introducing logical vocabulary that no new inferences involving only the old vocabulary be made appropriate thereby.
>
> (Brandom 2000, 68–69)

The distinctive feature of logical vocabulary, on Brandom's view, is that it makes explicit aspects of a community's inferential practice. According to Brandom, the conceptual content of nonlogical expressions is determined by the material inferences they license. Logical expressions serve to make explicit this implicit content. So if the addition of new logical vocabulary results in new inferences in the old vocabulary becoming valid, then the inferential connections of the old vocabulary have been altered.

Even if conservativeness can be motivated as a constraint on logical vocabulary, it has limitations. One issue is that conservativeness is a *global property* of a logical system. Consequently, whether an extension is conservative can depend on what other connectives are already in the system. As a result, logical inferentialists have attempted to impose a local property on inference rules that prohibit problematic connectives like tonk. Following Dummett, such a local constraint is often called *harmony*. Harmony, whatever the formal details, is a local property that holds, when it does, in virtue of the rules governing a given connective.[2] As recent work has shown, however, there are distinct concepts of harmony, even in the work of Dummett, and not all of these imply that certain additions will be conservative.[3]

A further point, on which we will focus, is that harmony itself has relevant parameters that can be brought out. These background or structural features of a logical system can represent aspects of an inferential practice, so they are of interest to a Dummettian-Brandomian logical inferentialist. In section 2, we will present some background on harmony, normalization, and cut elimination. Once we have this background in place, our discussion will have two streams: dropping structural rules of logical systems and enriching logical systems with additional structural elements. In section 3, we will consider substructural logics and the issues they bring for conservative extensions and harmony. The substructural setting introduces a bifurcation of logical connectives, leading to a well-known problem, namely the failure of a distributive law relating one form of conjunction and disjunction. We will present this issue in section 4 and consider plausible inferentialist responses, leading to the suggestion of enriching the sequent structure. Enrichment of sequent structure will be further considered in section 5 as a general response to problems with harmony and modal operators. Let us turn to the background.

2. Conservativeness

We will begin by discussing the conditional. One reason for this focus is that the conditional is Brandom's paradigm of logical vocabulary.

> Conditional claims—and claims formed by the use of logical vocabulary in general, of which the conditional is paradigmatic for the inferentialist—express a kind of semantic self-consciousness because they make explicit the inferential relations, consequences, and contents of ordinary nonlogical claims and concepts.
>
> (Brandom 2000, 21)

Consider the standard *natural deduction* introduction and elimination rule for the conditional:[4]

$$\dfrac{\begin{matrix}[A]^u \\ \vdots \\ B\end{matrix}}{A \to B}\;(u)(\to I) \qquad\qquad \dfrac{A \to B \quad A}{B}\;(\to E)$$

The leftmost rule is a hypothetical rule: If we can infer B from the assumption A, we can conclude that $A \to B$ and discharge the assumption A. The rule is also known as conditional proof. In Brandomian terms, it encapsulates the idea that the conditional $A \to B$ *makes explicit* an inferential connection between A and B. Correspondingly, the rightmost rule $(\to E)$ (*Modus Ponens*) allows us to infer B from A and $A \to B$. There is an intuitive harmony between the inference rules. The elimination rule allows us to infer B from A, thus unpacking the inferential connection.

The question is how we can make the intuitive idea of harmony between inference rules precise. What is it in general that is required for the introduction rules and elimination rules of a logical expression to be in harmony? One preliminary way of spelling it out is by invoking what Prawitz (1965) calls *reduction conversions*. Consider the following derivation (where Π_0, Π_1 are subderivations):

$$\dfrac{\dfrac{\begin{array}{c}[A]^1\\ \Pi_0\\ B\end{array}}{A \to B}\,{\scriptstyle (1)} \qquad \begin{array}{c}\Pi_1\\ A\end{array}}{B}$$

The above derivation has an application of ($\to I$) immediately followed by an application of ($\to E$). Prawitz notes that such a derivation can be converted into a derivation that avoids a detour:

$$\dfrac{\dfrac{\begin{array}{c}[A]^1\\ \Pi_0\\ B\end{array}}{A \to B}\,{\scriptstyle (1)} \qquad \begin{array}{c}\Pi_1\\ A\end{array}}{B} \quad \rightsquigarrow \quad \begin{array}{c}\Pi_1\\ A\\ \Pi_0\\ B\end{array}$$

Since the original derivation already had a derivation of A, and a derivation of B from A, the detour through the applications of ($\to I$) and ($\to E$) is unnecessary. This is the reduction conversion for the conditional.

The existence of the conversion is a decisive component in the proof of normalization. Suppose the natural deduction system only has the inference rules ($\to I$) and ($\to E$). Then we can show that every derivation can be transformed into a corresponding derivation in *normal form* (i.e. a derivation without any formula occurrence $A \to B$ that is both the conclusion of an ($\to I$) application and the major premise in a ($\to E$) application). That in turn leads to the *sub-formula property*, the fact that if A is derivable from Γ, there is a derivation where every node is a subformula of A or some $B \in \Gamma$. Normalization and the subformula property can also be proved if we add the standard introduction and elimination rules for $\wedge$ and $\vee$. The subformula property then furthermore entails the *separation property*: if A is derivable from Γ, there is a derivation that only applies inference rules for connectives occurring in A or some $B \in \Gamma$. So, extending a language consisting of some members of $\{\vee, \wedge, \to\}$ to the language $\mathscr{L}^{\vee,\wedge,\to}$, and extending the proof system with the corresponding rules, always yields a conservative extension.

Conversion reductions are, in other words, a way of ensuring that the addition of standard connectives will be conservative extensions. It is natural to think that the idea can be generalized to a template for harmony as a local constraint on inference rules. In turn, the formal harmony constraint should entail conservativeness. Harmony can then serve as a recipe for how to construct inference rules for new logical vocabulary in a way that does not alter the conceptual content of the original vocabulary. Although the idea works well for a limited set of connectives, it runs into trouble when extended to more interesting languages.

One reason is that there are connectives that have reduction conversions but whose introduction leads to nonconservativeness. Consider for example the following connective, bullet:[5]

$$\dfrac{\begin{array}{c}[\bullet]^u\\ \vdots\\ \bot\end{array}}{\bullet}\,(u)(\bullet I) \qquad \dfrac{\bullet \quad \bullet}{\bot}\,(\bullet E)$$

With the rules $(\bullet I)$ and $(\bullet E)$ we can give the following reduction conversion:

$$\dfrac{\dfrac{\begin{array}{c}[\bullet]^1\\ \Pi_0\\ \bot\end{array}}{\bullet}\,(1) \quad \begin{array}{c}\Pi_1\\ \bullet\end{array}}{\bot} \quad \rightsquigarrow \quad \begin{array}{c}\Pi_1\\ \bullet\\ \Pi_0\\ \bot\end{array}$$

Nonetheless, $(\bullet I)$ and $(\bullet E)$ leads to inconsistency in a straightforward manner (i.e., it allows a derivation of $\bot$).

$$\dfrac{\dfrac{\dfrac{[\bullet]^1 \quad [\bullet]^1}{\bot}}{\bullet}\,(1) \qquad \dfrac{\dfrac{[\bullet]^2 \quad [\bullet]^2}{\bot}}{\bullet}\,(2)}{\bot}$$

To apply the reduction conversion, one branch, say the leftmost, needs to be pasted onto each of the discharged assumptions of the other branch. It is not too hard to see that repeated applications of the reduction conversion will not terminate. So, the existence of a reduction conversion does not guarantee conservativeness. Further, given the usual rules governing $\bot$, this then leads to triviality: the addition of bullet, governed by the rules $(\bullet I)$ and $(\bullet E)$ results in every formula being provable.

It is tempting to blame the problem on the relative strangeness of a connective like $\bullet$. Not unlike tonk, it is designed to lead to inconsistency. Perhaps there are other conditions the inferentialist could apply in order to block connectives of this sort. But unfortunately there are other, less artificial connectives that also resist an analysis in terms of reduction conversions.

Negation is a case in point. The good news is that intuitionistic negation can also be conservatively added to the previous language with $\vee$, $\wedge$, and $\rightarrow$. Take the following standard introduction and elimination rule for the intuitionistic negation:

$$\dfrac{\begin{array}{c}[A]^u \\ \vdots \\ \bot\end{array}}{\neg A}\,(u)(\neg I) \qquad \dfrac{\neg A \quad A}{\bot}\,(\neg E)$$

The rules give rise to a conversion reduction just like the one we saw for the conditional above:

$$\dfrac{\dfrac{\begin{array}{c}[A]^1 \\ \Pi_0 \\ \bot\end{array}}{\neg A}\,(1) \qquad \begin{array}{c}\Pi_1 \\ A\end{array}}{\bot} \quad \rightsquigarrow \quad \begin{array}{c}\Pi_1 \\ A \\ \Pi_0 \\ \bot\end{array}$$

The resulting proof system has the separation property. Unfortunately, this is where the good news ends. Although the separation result holds for intuitionistic negation, it cannot be straightforwardly extended to classical negation.

The standard inference rules for classical negation do not have a reduction conversion.[6] Indeed, when classical negation is added to the negation free fragment we get a nonconservative extension. In the presence of classical negation, we can derive theorems in the conditional language that aren't derivable without the negation rules. The most famous example is Peirce's Law, $((A \rightarrow B) \rightarrow A) \rightarrow A$. In fact, this is crucial for the formalization of classical logic, as without classical negation, the inference rules $(\rightarrow I)$ and $(\rightarrow E)$ only give us the weaker intuitionistic conditional.

Put differently, the natural deduction rules for classical logic appear to violate the spirit of logical inferentialism. The introduction and elimination rules for the conditional aren't alone sufficient to prove the class of theorems in the $\rightarrow$-fragment of the language. That is, they aren't sufficient to determine the conceptual content of the classical conditional. For that classical negation is required.

Dummett, Prawitz, and other intuitionists have argued that the nonconservativeness of classical negation shows that the classical inference rules fail to fix a conceptual content at all. Classical negation, they conclude, is semantically dysfunctional. That leads to a revisionary inferentialism where the harmony constraint points in favor of intuitionistic negation. However, classical logicians have pointed out a number of problems with the revisionary argument. First, the argument relies on the assumption that the conceptual content of a logical expression is determined by the inference rules for that connective alone, what Paoli calls operational meaning. In contrast, an inferentialist who is willing to accept a form of semantic holism could accept that connectives only acquire their content in the context of a full system of logical connectives, Paoli's global meaning.[7] Second, even if classical negation fails to satisfy the harmony constraint, it doesn't follow that intuitionistic negation is the only alternative. And third, it turns out that the nonconservativeness of classical negation depends on formal properties of the proof system in question. This third objection points to a more severe limitation with nonconservativeness as a constraint on logical expressions.

In a sequent calculus system, the inference rules for connectives is given in a multiple conclusion form. An operational rule in sequent calculus has a finite set of premise sequents $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$ and a conclusion sequent $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite multisets of formulas. The operational rules for the classical conditional and negation can then be presented as follows:

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma', B \Rightarrow \Delta'}{\Gamma, \Gamma' A \rightarrow B \Rightarrow \Delta, \Delta'}\ (L\rightarrow) \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightarrow B, \Delta}\ (R\rightarrow)$$

$$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}\ (L\neg) \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}\ (R\neg)$$

It is easy to see that in the presence of the standard structural rules (identity, weakening, contraction), $(L\rightarrow)$ and $(R\rightarrow)$ are sufficient to prove Peirce's Law.[8] In fact, every classical consequence in the $\rightarrow$-fragment is derivable using the rules.

In the sequent calculus system, there is a counterpart to the reduction conversions and normalization of natural deduction. One shows that the conclusion of a proof that uses the cut rule,

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}\ (cut)$$

can be obtained in a way that does not use the cut rule. Let us say that a rule $\mathcal{R}$

$$\frac{S_1, \ldots, S_n}{S}\ (\mathcal{R})$$

is admissible in a sequent system if and only if: the premiss sequents $S_1, \ldots, S_n$ are derivable, then the conclusion is derivable. The standard proofs that cut is admissible rely on the following type of reduction conversions, which are the counterparts of reduction conversions:[9]

$$\dfrac{\dfrac{\begin{matrix}\Pi \\ \Gamma, A \Rightarrow \Delta\end{matrix}}{\Gamma \Rightarrow \neg A, \Delta} \quad \dfrac{\begin{matrix}\Pi' \\ \Gamma' \Rightarrow A, \Delta'\end{matrix}}{\Gamma', \neg A \Rightarrow \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\Pi' \\ \Gamma' \Rightarrow A, \Delta'\end{matrix} \qquad \begin{matrix}\Pi \\ \Gamma, A \Rightarrow \Delta\end{matrix}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Showing that cut is admissible in a sequent system without it is a standard technique for obtaining conservative extension results for sequent systems. Cut admissibility, or elimination, theorems are the sequent analogues of normalization theorems in the natural deduction setting. Cut is distinguished in many sequent systems as the only rule in which a connective disappears. If a sequent that is derivable with cut is derivable without cut, then it is derivable using rules in which formulas and connectives do not disappear, moving from premiss to conclusion. The cut admissibility theorem entails the separation property as a corollary. So, in a multiple conclusion sequent system, the addition of classical negation with its standard operational rules is a conservative extension of the →-fragment, unlike in the natural deduction case.

It should be clear that conservativeness depends on the choice of proof system. There are logical expressions whose addition in a sequent calculus yields a conservative extension but whose addition in natural deduction, with corresponding rules, yields a nonconservative extension. What is more, by allowing multiple conclusions in natural deduction, it is also possible to formalize classical negation and conditional with the separation property.[10]

This raises an important question about which formal framework for inference rules best captures a given inferential practice. It is true that introducing sets (or multisets) of conclusions is a further complication, but it is not one that lacks a philosophical interpretation. In fact, Restall (2005) has provided an account of multiple conclusion sequents that is potentially a good fit with logical inferentialism. On this interpretation, derivable sequents provide information about norms of assertion and denial: a derivable sequent

$$A_1, \dots, A_n \Rightarrow B_1, \dots, B_m$$

says that one cannot coherently assert all the antecedent formulas while denying all the succedent formulas. We then get corresponding interpretations of the operational rules presented above. The rule ($R\neg$), for example, encodes a norm saying that if one cannot assert all the members of Γ and A while simultaneously denying every member of Δ, then one cannot assert all the members of Γ while simultaneously denying $\neg A$ and every member of Δ. If we simplify by dropping the auxiliary formulas, it says that if you cannot assert A, then you cannot deny $\neg A$.

While some philosophers have objected to the use of multiple conclusion sequent systems (Dummett 1991, 40–41; Tennant 1997, 319–320; and Steinberger 2011b), we think that the inferentialist still has reason to adopt them.[11] Additionally, although much of the focus of Brandom (1994) is on *assertion*, including both assertion and denial in the description of the inferential role of expressions can accommodate much of Brandom's view.

The stakes for the inferentialist are greater than just the conservativeness of classical negation. Many connectives are only conservative given certain assumptions about the structural properties of the proof system. Once the inferentialist can vary the structural properties of the proof system to capture different inferential practices, well-behaved connectives might prove problematic, and ill-behaved connectives might become legitimate.

3. Practice and Parameters

For a second example of structural properties, we can stick with the conditional. Recall the standard inference rules for the intuitionistic conditional, $(\to I)$ and $(\to E)$. There are structural properties built into the $(\to I)$ rule we would like to highlight. The rule is hypothetical—that is, it allows us to discharge assumptions. Hypothetical rules are governed by discharge policies, in particular that in an application of $(\to I)$, one discharges 0 or more occurrences of an assumption A. That leads to two special cases of $(\to I)$, vacuous and multiple discharge respectively.

$$\frac{B}{A \to B} \qquad \frac{\begin{array}{c}\underbrace{[A]^u \ldots [A]^u}\\ \vdots \\ B\end{array}}{A \to B}\,(u)$$

In the leftmost derivation, $(\to I)$ is applied without discharging any copies of the antecedent A, while the rightmost derivation discharges multiple copies of A. These special discharge policies are not logically idle. Without the former we cannot derive $A \to (B \to A)$ in the $\to$-fragment, and without the latter we cannot derive $(A \to (A \to B)) \to (A \to B)$. It should be clear, therefore, that $(\to I)$ and $(\to E)$ only axiomatize the intuitionistic conditional if the discharge policies are permitted.[12]

The reduction conversions are also affected by the presence of the special discharge policies.[13] Suppose that the proof Π_0 in the unreduced proof example above has two occurrences of A that are discharged by $(\to I)$. Then, in the reduced proof, there will be two copies of Π_1 in the resulting proof, one for each assumption of A that is being replaced.

We said that the discharge policies were implicit in the natural deduction system. They are implicit in the sense that they are not notationally marked. For our purposes, it will be easier to work with sequent calculus systems, in particular the multiple conclusion rules ($L\rightarrow$) and ($R\rightarrow$) displayed above. In addition to these rules and the axioms, $A \Rightarrow A$, standard sequent calculus systems have *structural rules* that correspond roughly to the discharge policies in natural deduction:[14]

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (LK) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}\ (RK)$$

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ (LW) \qquad \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}\ (RW)$$

The topmost rules—antecedent and succedent weakening—correspond to vacuous discharge of assumptions, while the bottommost rules—antecedent and succedent contraction—correspond to multiple discharge of assumptions.[15]

These structural rules reflect features of inferential practices, and there are practices in which these rules may not be appropriate.[16] An inferential practice in which the rules (RK) and (LK) do not reflect the structure of the practice is suggested in the motivating comments of relevant logicians, such as Anderson and Belnap (1975).[17] A community might require that, in a good argument, all the premises are used in a substantive way in obtaining the conclusion(s). There can be no idle premises. If we are trying to formalize the inferential practice of this community, we should not use the weakening rules, which permit inferences rejected by the community.[18]

Similarly, there are examples of inferential practices that, arguably, reject the rule of contraction. One example is the geometers of the early twentieth century.[19] The geometers were concerned to mark how many times they appealed to certain assumptions, evaluating arguments differently depending on how many times the assumptions were used. It is a small step of idealization to a community that requires that whenever an argument uses an assumption multiple times, the assumption has to be made multiple times, which is to say that they reject the structural rules (LW) and (RW). Another example is supplied by Barker (2010). Barker argues that the phenomenon of free choice permission is best understood by appeal to the distinction between connectives that emerges when contraction is dropped. If he is right, then contemporary English speakers, at least sometimes, engage in inferential practices best formalized without (LW) and (RW).

These examples show how the structural rules reflect aspects of an inferential practice. Different practices may require the adoption or rejection of different sets of structural rules.[20] An inferential practice need not manifest the structural rules in an overt form. They may remain implicit in the practice, in the sense that no one explicitly engages in an inference whose form matches that of the rule. Still, an explicit regimentation of the practice may contain the rules. Since the presence or absence of the structural rules results in different logics, the different conditionals make explicit different sorts of inferential practices.

One familiar consequence of dropping weakening and contraction is that classically equivalent operational rules become distinct. To stick with the conditional, consider the two following implicational connectives $\sqsupset$ and $\multimap$.

$$\dfrac{\Gamma \Rightarrow A,\Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \sqsupset B \Rightarrow \Delta}\ (L\sqsupset) \qquad \dfrac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow A \sqsupset B, \Delta}\ (R\sqsupset_i) \qquad \dfrac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \sqsupset B, \Delta}\ (R\sqsupset_{ii})$$

$$\dfrac{\Gamma \Rightarrow A, \Delta \qquad \Gamma', B \Rightarrow \Delta'}{\Gamma, \Gamma', A \multimap B \Rightarrow \Delta, \Delta'}\ (L\multimap) \qquad \dfrac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \multimap B, \Delta}\ (R\multimap)$$

The connectives $\sqsupset$ and $\multimap$ are equivalent in the presence of weakening and contraction. Indeed, they are just two variants of the classical conditional. However, once we drop either of the structural rules, they come apart. A similar bifurcation happens for conjunction and disjunction. In general, the substructural logics allow for distinct context-sharing (additive) and context-independent (multiplicative) connectives.

The substructural systems also produce nonconservative extensions with connectives that are conservative in the fully structural systems. Certain connectives reintroduce structural rules, thereby allowing the derivation of new sequents in the original language. An example is conditionals defined by mixing the context-sharing and context-independent operational rules above. Suppose we have a system with $\sqsupset$ and $\multimap$, but without contraction. We then extend the system with a new conditional, $\rightharpoondown$:

$$\dfrac{\Gamma \Rightarrow A,\Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \rightharpoondown B \Rightarrow \Delta}\ (L\rightharpoondown) \qquad \dfrac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \rightharpoondown B, \Delta}\ (R\rightharpoondown)$$

The new conditional $\rightharpoondown$ has the left-rule of $\sqsupset$ but the right-rule of $\multimap$. The result is that, provided we have (LK) and (RK), we can derive a restricted version of (LW) where there is at least one succedent formula (B below):

$$\dfrac{\dfrac{\dfrac{A, A \Rightarrow B}{A \Rightarrow A \rightharpoondown B}}{\Rightarrow A \rightharpoondown (A \rightharpoondown B)} \qquad \dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A, B} \quad \dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A, B} \quad \dfrac{B \Rightarrow B}{A, B \Rightarrow B}}{A, A \rightharpoondown B \Rightarrow B}}{A, A \rightharpoondown (A \rightharpoondown B) \Rightarrow B}}{A \Rightarrow B}$$

The presence of the restricted contraction rule will affect which sequents are derivable for the other conditionals. For example, suppose we have a sequent system with $\multimap$ and the weakening rules, but not the contraction rules. In the extension with $\rightharpoondown$, we can derive the sequent $\Rightarrow (A \multimap (A \multimap B)) \multimap (A \multimap B)$, which was underivable in the original system.

Similar nonconservative extensions with weakening can be produced by combining ($R\sqsupset$) and ($L\multimap$). The result is a conditional, $\rightharpoonup$, that allows the reintroduction of a restricted version of weakening.[21]

Next, let us look again at Read's connective bullet, •. The rules for bullet normalize, as Read indicates. The addition of bullet to, say, intuitionistic logic would result in inconsistency and so nonconservative extension. However, the derivation of inconsistency crucially uses contraction, or multiple discharge. It could be added to a non-contractive system in a conservative way. This is an instance of a broader point, made by Restall (2010), that the context of deducibility, which encompasses the structural rules, has far-reaching consequences for logical issues, such as conservative extension and definability. In substructural logics, the context of deducibility is modified from that of classical and intuitionistic logic.

One example that is sometimes held up as a counterexample to the claim that harmony implies conservative extension is the truth predicate.[22] While not a *connective*, there is some reason to think that the truth predicate is logical, accorded distinguished status along with identity, and the rules

$$\frac{A}{T\langle A\rangle}\ (TI) \qquad \frac{T\langle A\rangle}{A}\ (TE)$$

are invertible and seem harmonious. Adding these rules to classical arithmetic, without any restrictions, results in inconsistency, and so nonconservative extension.[23] In a noncontractive logic such as *RW*, one can add these rules without restriction to arithmetic nontrivially.[24] As far as we know, the question of whether the extension is conservative remains open.[25]

Next, we will return to the example with which we started, tonk.

$$\frac{A, \Gamma \Rightarrow \Delta}{A \bowtie B, \Gamma \Rightarrow \Delta}\ (L\bowtie) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, B \bowtie A}\ (R\bowtie)$$

In the presence of all the structural rules, neither bullet nor tonk can be added conservatively. As we have seen above, however, the inferentialist can salvage bullet through rejection of the structural rule of contraction. The question, then, is whether something similar is possible for tonk. In fact, there is another structural rule to discuss, namely *cut*. Ripley (2015) has argued that tonk is acceptable to the inferentialist provided that cut is rejected. Indeed, the addition of tonk to a system whose only rules are contraction and weakening, together with the axioms, will be conservative, although cut will not be admissible.

As with the other structural rules, cut corresponds to features of inferential practices. Cut is sometimes described as encoding the use of lemmas in proofs. Something is proved once and cited as needed, without reproducing the proof of the lemma in the course of the reasoning. The inferential practice of classical mathematics, on one construal, would be well formalized using the cut rule.

Another example is supplied by Restall (2005). Taking the rule contrapositively, Restall says, "It tells us that if *A* is undeniable in the context of [coherently asserting Γ and denying Δ] then it is coherent to assert *A*, provided that [asserting Γ and denying Δ] is already coherent."[26] On Restall's view, cut codifies a certain sort of coordination of assertions and denials.[27] If, by the rules of logic, *A* is undeniable in a given context, then it is coherently assertible in that context, and if *A* cannot be coherently asserted in that context, then it is undeniable.

One can imagine inferential practices in which assertion and denial are not coordinated in the way that Restall describes. Ripley motivates the failure of cut by appeal to the same bilateralist interpretation of sequents as Restall. On Ripley's view, there is more leeway between assertion, denial, and claims that are out of bounds than on Restall's view.[28] A claim being undeniable as a matter of logic in a given, coherent context does not thereby mean the claim is coherently assertible in that context, and similarly, an unassertible claim is not forced to be deniable. There can be gaps between the assertible and the undeniable and between the undeniable and the unassertible. On this view, assertion and denial each split into strict and tolerant forms, which interact in interesting ways. It appears, then, that the presence or absence of cut, as admissible or primitive, also codifies different norms at play in inferential practices.[29]

We will close this section echoing a point made by Ripley (2013b). If all one wants in one's choice of sequent system is a guarantee of conservative extension, then cut should not be taken as a primitive rule. Provided that formulas from the premises of a rule do not disappear in the conclusion, then extension of the system with rules for new connectives will be conservative, since any newly derivable sequent will have an occurrence of the new connective in it. If formulas disappear in the conclusion of a rule, however, one will not have this assurance, as demonstrated by Wansing's super-tonk.[30] Admissible rules may not be preserved under extension, whereas a primitive rule of a system will be preserved. When cut is not a primitive rule, one may be in the position of having cut admissible in the system prior to extension but no longer admissible post-extension. Whether the loss of cut is a major problem will depend on one's interpretation of sequents and views about logical consequence, such as whether cut is a primitive rule of the system, and whether cut admissibility is a part of the harmony constraint for sequent systems. We bring up these questions to emphasize that cut is another parameter of inferential practice that should be specified when considering what one wants in harmony. We now turn to issues of proof-theoretic structure.

4. Distribution and Structure

Classical negation provides motivation to adopt multiple conclusion sequents, as it is difficult to add to a single-conclusion system with a conditional in a harmonious way.[31] Multiple conclusions can be seen as an enrichment of the single conclusion sequent structure. There are other forms of structure with which one can enrich sequents. In this section, we will motivate one sort of enrichment in a substructural setting.

In the previous section, we presented context-sharing (additive) and context independent (multiplicative) forms of rules for implication. The distinction extends to other connectives, and we present the rules for conjunction here, the disjunction rules being straightforward duals.

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \circ B \Rightarrow \Delta}\ (L\circ) \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Sigma \Rightarrow B, \Theta}{\Gamma, \Sigma \Rightarrow A \circ B, \Delta, \Theta}\ (R\circ)$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ (L\wedge) \quad \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ (L\wedge) \quad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}\ (R\wedge)$$

In intuitionistic and classical logic, where we have all the structural rules, the additive and multiplicative are equivalent, in the sense that both

$$A \circ B \Rightarrow A \wedge B \text{ and } A \wedge B \Rightarrow A \circ B$$

are derivable.[32] When either of weakening or contraction is dropped, the two connectives come apart, in the sense that one of the two sequents will not be derivable.

Consider a community that uses a conditional, but does not permit vacuous introduction and so rejects weakening. The community encounters conjunction and disjunction, understood according to their additive rules, and adopts them. A sequent system that represents their inferential situation uses the conditional, additive forms of conjunction and disjunction, with the rules above, and contraction, but not weakening. The resulting system is known to have cut admissibility, but the law of distribution is not derivable.[33]

$$A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$$

This system yields the positive fragment of the relevant logic *R* minus distribution.[34]

There is an intuition that if additive disjunction and conjunction are to mean the same thing in the substructural setting as in the classical, then they should obey distribution. After all, the additive rules yield distribution in the presence of contraction and weakening. This intuition has motivated some to add a primitive rule of distribution.[35]

$$\frac{A \wedge (B \vee C)}{(A \wedge B) \vee (A \wedge C)}\ (Dist)$$

It is not clear that this is open to the inferentialist. It can spoil normalization, and it does not fit neatly into the dichotomy of introduction and elimination rules, while clearly not being a structural rule. We will briefly look at three ways of responding to this issue.

The first is to maintain that the additive rules alone determine the meaning of conjunction and disjunction, and that is insufficient to secure distribution. So, additive conjunction and disjunction do not obey distribution. The fact that they distribute in classical logic is a side effect of the structural rules, rather than the meaning determined by the operational rules.[36] While this view is open to the inferentialist, we will set it aside to look at some options that maintain the intuition that additive conjunction distributes over additive disjunction.

An alternative response begins by noting that the problem with deriving distribution is that one is restricted as to what additional assumptions are available for use in the ($L\vee$) step, or ($\vee E$) in natural deduction. The response then is to adopt a natural strengthening of the additive disjunction elimination rule.[37] According to this rule, one is permitted to freely use side premises in the subproofs for ($\vee E$) that the major premiss disjunction depends upon.[38]

This system enjoys normalization and permits the derivation of distribution. The strengthened form, however, results in a logic properly stronger than (positive) R but still weaker than classical logic. The difference between the two rules is whether they permit the use of formulas in ($\vee E$) that depend on the same assumptions as the disjunction. In classical logic, there are no restrictions on what side premises are used in ($\vee E$), but in the substructural setting the added flexibility matters.

The third response takes the view that distribution should be derivable for the target vocabulary and it secures this by enriching the sequent system with additional structural connectives governed by their own structural rules.[39] In the basic sequent systems we have been discussing, the comma is the only structural connective. Once the distinction between additive and multiplicative connectives becomes important, as in substructural logics, it is natural to see them as reflecting different structural features of inferential practices. Let us then add to the sequent system a structural connective, ";", in addition to the comma. The two structural connectives can be governed by different sets of structural rules. In particular, the comma, but not the semicolon, obeys (LK) and (RK), and both obey (LW) and (RW). The connective rules are changed to reflect the differing roles of these structural elements, with the conditional and multiplicative conjunction going with semicolon and conjunction going with comma, as in the following examples.[40]

$$\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A;B \Rightarrow A \circ B}\,(R\circ)}{A \circ B \Rightarrow A \circ B}\,(L\circ) \qquad \dfrac{\dfrac{\dfrac{A \Rightarrow A}{A,B \Rightarrow A}\,(LK)}{A \wedge B, A \wedge B \Rightarrow A}\,(L\wedge)}{A \wedge B \Rightarrow A}\,(LW)$$

A sequent system for positive R using two structural connectives enjoys cut admissibility and permits the derivation of distribution.[41]

The question this option raises for the inferentialist is whether the new structural element can be understood in terms of inferential practices. The two structural elements are different ways of combining premises (or conclusions).[42] The semicolon is an intensional way, appropriate when one is extracting information from a conditional via *modus ponens*. The comma, on the other hand, is extensional, appropriate for simply pooling information from some premises.[43] The inferentialist can understand these structural connectives in terms of practices, and their addition can secure some of the desired formal properties, such as cut admissibility, for certain logics.

The third response uses an idea that we will discuss further, namely enriching the basic proof-theoretic structure. For the remaining discussion, we will turn to systems for classical logic, focusing on extensions with modal vocabulary.

5. Modality and Structure

We have, so far, considered systems in which some of the fundamental structural rules have been dropped. These properties are important for normalization, cut admissibility, conservative extension, and harmony. In this section, we will look at enriching sequent systems with additional structure.

Modal connectives provide initial motivation for additional structure. There are a wide variety of modal logics, but it is notoriously difficult to provide satisfactory proof systems for them. One recalcitrant example is the modal logic S5. One of the first sequent systems for it was provided by Ohnishi and Matsumoto (1957). The modal rules are the following.

$$\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ (L\Box) \qquad \frac{\Box\Gamma \Rightarrow \Box\Delta, A}{\Box\Gamma \Rightarrow \Box\Delta, \Box A}\ (R\Box)$$

In the $(R\Box)$ rule, $\Box\Gamma$ is $\Box A_1, \ldots, \Box A_n$, where Γ is $A_1, \ldots, A_n$. Ohnishi and Matsumoto point out that certain sequents are only derivable using cut. Their $(R\Box)$ rule has a strong restriction on the form of side formulas, which requires one to introduce a box at one step in the derivation and then use cut on a side formula to insert a side formula that does not have the appropriate form.

Further, as Read points out, the natural deduction introduction and elimination rules for the box

$$\frac{A}{\Box A}\ (\Box I) \qquad \frac{\Box A}{A}\ (\Box E)$$

and diamond

$$\frac{A}{\Diamond A}\ (\Diamond I) \qquad \frac{\Diamond A \quad \begin{matrix}[A]^u \\ \Pi \\ B\end{matrix}}{B}\ (u)(\Diamond E)$$

are intuitively not in harmony.[44] While the ($\Box E$) rule permits the transition from $\Box A$ to A, ($\Box I$), in S4 and S5, does not let one move from A to $\Box A$ unless certain side conditions are satisfied. In Prawitz's formulation, the side condition is that the open assumptions on which the premise of ($\Box I$) depends must be appropriately modal. The minor premiss of the rule ($\Diamond E$) is subject to a similar restriction as ($\Box I$), namely, that all assumptions of the subproof, apart from the assumed A, are appropriately modal.

Although the S4 rules ($\Box I$) and ($\Box E$) normalize and their addition to classical logic is conservative, they are not in harmony.[45] Both S4 and S5 have the same rules for the modal operators, differing only in the side conditions. As Read (2008) puts the point, we should expect that since the logics are different, the rules should differ. The ($\Box E$) rule in harmony with the ($\Box I$) would be one weakened to reflect the side condition on ($\Box I$). On the possibility side, the ($\Diamond E$) rule in harmony with the ($\Diamond I$) rule would be one without a side condition, but this rule trivializes the modality, as well as being insensitive to the differences between S4 and S5, which share a ($\Diamond I$) rule.

A natural diagnosis of the issue that arises both with natural deduction and sequent presentations of modal logic is that there are not enough parts to the basic structure, whether formulas or sequents, variations of which permit the distinctions needed by the different modal operators. A suggestion from Read is to enrich the system with additional structure, namely labels for worlds, $A : i$, and special formulas relating worlds, Rij.[46] The rules for $\Diamond$ then become the following.

$$\frac{A : j \quad Rij}{\Diamond A : i}\ (\Diamond I) \qquad \frac{\Diamond A : i \quad \begin{matrix}[Rij, A : j] \\ \Pi \\ B : k\end{matrix}}{B : k}\ (u)(\Diamond E)$$

In the ($\Diamond E$) rule, $i \neq j \neq k$, and no assumptions of minor premiss subproof are labelled with j besides the displayed ones. The rules for $\Box$ can be adjusted similarly. These rules normalize. The distinction between S4 and S5 is brought out by the different rules added to the system for the Rij formulas.

One can add labels to sequents, along with special formulas indicating relations between the labels.[47] As in the natural deduction system, different modal logics are accounted for using structural rules on the relational formulas. As an alternative to labels, one can enrich sequents in the direction of *hypersequents*, which we will now discuss.[48] A hypersequent,

$$X_1 \Rightarrow Y_1 \mid \ldots \mid X_n \Rightarrow Y_n,$$

is a multiset of sequents. Restall (2007) provides a hypersequent system for S5 that enjoys cut elimination.[49]

For the move to hypersequents to be appealing to the Brandomian, there needs to be a feature of the inferential practice that, in some sense, corresponds to the proposed structure. Such a feature has been supplied by Restall. As we saw above, multiple conclusion systems can be understood in terms of norms of assertion and denial. The additional hypersequent structure can be understood in terms of consideration of alternatives.[50] Derivable hypersequents present norms governing assertion and denial in alternative situations.

The necessity rules are the following.

$$\frac{\mathcal{H}\left[\Sigma \Rightarrow \Theta \middle| A, \Gamma \Rightarrow \Delta\right]}{\mathcal{H}\left[\Box A, \Sigma \Rightarrow \Theta \middle| \Gamma \Rightarrow \Delta\right]}\ (L\Box) \qquad \frac{\mathcal{H}\left[\Gamma \Rightarrow \Delta \middle| \Rightarrow A\right]}{\mathcal{H}[\Gamma \Rightarrow \Delta, \Box A]}\ (R\Box)$$

In these rules, $\mathcal{H}$ is a hypersequent that contains the components displayed in the brackets. Cut is admissible in this system, unlike that of Ohnishi and Matsumoto (1957). The system has the subformula property, so the addition of the S5 modalities to classical logic is a conservative extension.

S5 is one of the standard philosophical modal logics. The inferentialist has, we think, strong reason to provide a treatment of S5 that is acceptable by her lights. This has proved difficult to do in the basic, unaugmented sequent setting.[51] It shows up as the distinguished modal logic of Brandom's incompatibility semantics.[52] Brandom says, "S5 accordingly has some claim to being *the* modal logic of consequence relations, whether material or logical."[53]

The added structure of hypersequents permits the codification of norms dealing with assertion and denial in various alternatives. Consideration of alternatives is natural in inferential practices. The necessity operator makes explicit claims whose assertion or denial affects coherence across alternatives. Some claims can be denied in certain alternatives, in combination with other assertions and denials. Some assertions and denials have a modal force that extends beyond a particular hypothetical situation.

There can be different sorts of consideration of alternatives, which naturally motivates a further enrichment of the hypersequent structure. As investigated by Restall (2012), one can extend the hypersequents with another type of hypersequent separator, representing a second dimension to the consideration of alternatives. These two-dimensional systems provide a representation of a practice of considering indicative and subjunctive alternatives.[54] Further operators can be introduced to make explicit features of the inferential practice represented by this richer system, such as *a priori* knowledge and actuality. These systems enjoy cut admissibility and so the addition of these operators is conservative.

A community's inferential practices may broadly concern not just inference and assertion, but also consideration of hypothetical alternatives, as well as being sensitive to considerations not covered here, such as time and explicit concern with deontic statuses. The basic sequent structure may be sufficient for representing features of a relatively simple inferential practice, but a richer practice may need a richer sequent structure to adequately make the features of the practice explicit. We now turn to our general conclusions.

6. Conclusion

The rules for tonk present a problem for the logical inferentialist: there needs to be a principled way to separate the acceptable combinations of rules from the unacceptable. Dummett's proposal, harmony between the introduction and elimination rules, offers a way to demarcate these collections, provided there is some further analysis of harmony. Brandom tentatively endorses Dummett's view, with respect to logical connectives, although it appears that harmony is merely a means to the end of securing conservative extensions.[55]

What we have argued is that the concept of harmony has several parameters that need to be settled to apply the concept. The very notion of a problem case depends upon this. As we saw above, classical negation appears to be a problem when single conclusion sequents and natural deduction are under consideration, but it is fine in multiple conclusion frameworks. Further, the connective bullet can be added conservatively in a contraction-free framework. Indeed, if cut is up for grabs, then even tonk can be embraced by the inferentialist. These observations point to a more general dynamic. If structural rules are dropped, then standard connectives that can be added conservatively when all structural rules are present may yield nonconservative extensions, depending on the formulation of the rules. Even the conditional, which sits at the heart of Brandomian inferentialism, makes explicit different material inferences depending on the structural rules.

The move to a substructural setting brings with it distinctions between previously equivalent connectives. We noted the well-known fact that a form of the distribution law for conjunction and disjunction is underivable with the additive rules. One response to this is to enrich the proof system with additional sequent structure. This move will be appealing to the inferentialist if one can understand the addition in terms of features of an inferential practice. The addition can be so understood: the additional structural connective captures an additional way to combine premises. Substructural logics can be sensitive to which premises were used in a derivation in a way that classical logic is not. The richer inferential setting of substructural logic motivates the addition of a richer sequent structure.

Mirroring the structure of inferential practice in sequent structure is not restricted to substructural logics. Modal logics naturally motivate an enrichment of natural deduction and sequent structure. Hypersequent structure can be understood in inferential terms, namely the consideration of alternatives. A richer inferential practice, one involving considerations of different alternatives, temporal relations, or explicit deontic or epistemic evaluation, could call for the use of additional or different sequent structure. The richer sequent structure suggests a path to making explicit the relevant features of the target inferential practice.[56]

Notes

1 The notation is our own.
2 Harmony is typically viewed as ensuring that the elimination rules do not outstrip the introduction rules. The converse relation, that the introduction rules do not outrun the elimination rules, is sometimes called *stability*. Stability and similar concepts have been the focus of much recent work on logical inferentialism. See Pfenning and Davies (2001), Jacinto and Read (2016), and Dicher (2016) for further discussion.
3 See Read (2000) and Steinberger (2011a).
4 See Gentzen (1935) and Prawitz (1965).
5 See Read (2000; 2010).
6 There are many ways of formalizing classical negation. One could add any of the following rules to the rules for intuitionistic negation:

$$\dfrac{}{A \vee \neg A}\ (LEM) \qquad \dfrac{\begin{array}{c}[\neg A]^u\\ \vdots\\ \bot\end{array}}{A}\ (u)(CRAA) \qquad \dfrac{\neg\neg A}{A}\ (DNE)$$

7 See Paoli (2003) for discussion of the distinction between global and operational meaning.
8 The structural rules will be introduced in the next section.
9 The proof of Gentzen (1935) uses these types of conversions with a strengthening of cut.
See Negri et al. (2001) or Bimbó (2015) for proofs using reduction conversions on cut.
10 Cf. Read (2000), and Francez (2014c).
11 For a treatment of assertion and denial in a single-conclusion natural deduction setting, see Smiley (1996), Rumfitt (2000), or Francez (2014b). See Humberstone (2000) for further discussion.
12 Although in the presence of other connectives, that might change. Consider for instance the following derivation.

$$\cfrac{\cfrac{\cfrac{\cfrac{[A]^2 \quad [B]^1}{A \wedge B}}{A}}{B \rightarrow A}\,(1)}{A \rightarrow (B \rightarrow A)}\,(2)$$

13 See Francez (2014a) for details on the reduction conversions involved when vacuous discharge is disallowed.
14 See Negri et al. (2001, chap. 8) for more on the correspondence.
15 Here we follow Restall (2000), and other logicians in the tradition of combinatory logic, in using the labels "K" and "W" for weakening and contraction, respectively.
16 See also Paoli (2002) for an overview of different motivations for dropping structural rules.
17 Lance and Kremer (1996) motivate a related logic of relevant commitment entailment, on Brandomian inferentialist grounds.
18 As an aside, we note that dropping structural rules, particularly weakening, has reper- cussions for incompatibility-based inferentialism. Suppose that one introduces a connective, $\bot$, into the language to mark when some premises are incompatible. A natural axiom for it to obey is the following: $\bot \Rightarrow \varnothing$. In the presence of (RK), $\bot$ entails everything. One can define a negation using this and the conditional: $\neg A$ is $A \rightarrow \bot$. The negation of A is then guaranteed to be incompatible with A, in the sense that the following will be derivable.

$$\frac{A \Rightarrow A \quad \bot \Rightarrow \bot}{A, \neg A \Rightarrow \bot}\,(L\rightarrow)$$

Cutting on the $\bot$ axiom followed by an application of (RK) yields that incompatible formulas, here contradictory formulas, entail everything, giving sets of incompatible premises a distinctive inferential role. This fact is exploited, under a different presentation by Brandom (2008) and Peregrin (2015). It is an underlying assumption of their approaches that incompatibility persists through the addition of premises, which is a form of weakening. In some substructural settings, such as the relevant logic R, one can adopt the view that the conditional makes explicit moves from premises to conclusions while denying that sets of incompatible premises have a distinctive, explosive role. A modified definition of incompatibility entailment is needed for such substructural settings.

In addition to weakening, cut and contraction are built into the structural assumptions of incompatibility semantics. See, for example, Brandom (2008, 137). If either of cut or contraction is unavailable, other complications may be needed, but we leave this open here.
19 The details are supplied by Pambuccian (2004).
20 There are further structural rules one can distinguish, such as permutation, by taking structures on each side of the sequent separator to be sequences or more general structures, but we will focus on contraction, weakening, and cut.
21 See Hirokawa (1996), Humberstone (2007), and Rogerson (2007) for more on rules that permit the derivation of structural rules in substructural contexts.

22 See Read (2000). For a dissenting view, see Steinberger (2011a).
23 The truth predicate needs a theory of syntax to supply sentence names, and we are here taking that theory to be arithmetic.
24 See Restall (1992) or Petersen (2000).
25 Depending on the rules governing the syntactic theory and negation, the addition of the truth predicate and the syntactic theory to pure logic may be nonconservative in a noncontractive logic.
26 Restall (2005, 196–197).
27 We are treating assertion and denial as aspects of an inferential practice, following Brandom (1994, 206), who says, "For asserting and inferring are two sides of one coin; neither activity is intelligible except in relation to the other."
28 See Cobreros et al. (2011) and Ripley (2013a, 143–144) for discussion.
29 One might, following French (2016), wonder about the identity sequents, $A \Rightarrow A$. After all, the starting points of derivations are, in a broad sense, structural features. If one thinks it is sometimes coherent to assert and deny the same sentence, then, given the interpretation of sequents in terms of assertion and denial, identity sequents will not generally be acceptable.

We note that French considers a different interpretation of sequents, due to Malinowski, than we consider here.
30 The rules for super-tonk, provided by Wansing (2006), are the following.

$$\frac{\Gamma \Rightarrow \Delta}{\Psi \Rightarrow \textit{super–tonk}} \qquad \frac{\Psi \Rightarrow \textit{super–tonk}}{\Gamma \Rightarrow \Delta}$$

For further discussion, see Ripley (2015).
31 Note that we are not claiming that it is impossible.
32 See, for example, French and Ripley (2015).
33 One can see the appeal to weakening in the following derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, B \Rightarrow A}\,(LK) \quad \dfrac{B \Rightarrow B}{A, B \Rightarrow B}\,(LK)}{A, B \Rightarrow (A \wedge B)}}{A, B \Rightarrow (A \wedge B) \vee (A \wedge C)} \quad \dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, C \Rightarrow A}\,(LK) \quad \dfrac{C \Rightarrow C}{A, C \Rightarrow C}\,(LK)}{A, C \Rightarrow (A \wedge C)}}{A, C \Rightarrow (A \wedge B) \vee (A \wedge C)}}{\dfrac{A, (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)}{\dfrac{A \wedge (B \vee C), A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)}{A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)}}}$$

34 See Thistlewaite et al. (1988) for details.
35 This is the route adopted by Anderson and Belnap (1975).
36 See Paoli (2007; 2014) and Hjortland (2014) for discussion.
37 See Prawitz (1965), Charlwood (1981), and Giambrone and Urquhart (1987). We omit the rule since it requires either a lengthy side condition or the use of subscripts on formulas, conventions for which would take more explanation than is justified by the point at hand.
38 We focus on the natural deduction system, since it is easier to give a feel for the rule strengthening. In the sequent system, the strengthening is implemented through the use of a restricted weakening rule that lessens the impact of the context-sharing features of the additive disjunction rules.
39 See Belnap (1982), Read (1988), or Slaney (1990) for examples.
40 We omit general statements of the rules, as that would require some further notational details that would take us a bit afield.
41 See Belnap et al. (1980) or Dunn and Restall (2002) for more.

42 This view is elaborated by Slaney (1990).
43 These structural elements can nest, so it would be more correct to say that they are ways of combining structures, or bunches, of formulas.
44 Read (2008).
45 See Prawitz (1965) and Medeiros (2006).
46 See Read (2008).
47 Negri (2005).
48 For discussion of the relative philosophical merits of the use of labelled formulas, see Poggiolesi (2009), Humberstone (2011, 111–112), and Read (2015).
49 Poggiolesi (2008) also provides a hypersequent formulation of S5. See Bednarska and Indrzejczak (2015) for a survey of the area.
50 Restall (2012).
51 See Poggiolesi (2011), especially chap. 1, for discussion.
52 See Brandom (2008, 141–175).
53 Brandom (2008, 139), the original is bolded for emphasis, which we omit here.
54 See Lance and White (2007) and Restall (2012) for more on the consideration of alternatives.
55 We say "tentatively," since Brandom registers some criticisms of Dummett's view. See, e.g., Brandom (2000, 72–76).
56 We would like to thank Greg Restall, Rohan French, and Kai Tanter for discussion and comments on earlier versions of the material. Shawn Standefer's research was supported by the Australian Research Council, Discovery Grant DP150103801.

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