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An incompleteness theorem for modal relevant logics

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Abstract In this paper, an incompleteness theorem for modal extensions of relevant logics is proved. The proof uses elementary methods and builds upon the work of Fuhrmann.

Routley-Meyer ternary relational frames have been used to provide adequate modelings for a range of substructural logics, in particular relevant logics. A feature of these frames, to be defined below, is that they have a distinguished set of points, the *normal points*, that in general contains more than one point. This has lead some logicians to consider the class of *reduced frames*, frames whose set of normal points is a singleton. For many relevant logics, and even irrelevant logics, reduced frames are adequate.[1]

One might expect that reduced frames would likewise be adequate for a range of modal extensions of relevant logics. Indeed, there was an early completeness result in that area.[2] Fuhrmann [12], however, proved that for some logics, those contained in the logic he calls R.KT4, no completeness result is forthcoming for reduced frames.[3] In this paper, I will extend and strengthen Fuhrmann's result, proving that for a wider class of modal relevant logics, reduced frames are inadequate.

In §1, I will provide the requisite background on Routley-Meyer ternary relational frames for relevant logics as well as presenting axiom systems for the logics to be discussed. In §2, I will briefly explain Fuhrmann's incompleteness theorem, highlighting a key lemma. Finally, in §3, I will prove an incompleteness theorem for a range of modal relevant logics and conclude with three observations about these results.

1 Background

The family of relevant logics is large.[4] The focus here will be on a few members of this family, namely the logics B, C, and R.[5] The logic B is the logic of all Routley-Meyer frames.[6] The logic C is highlighted by Routley et al. [36, 289] as it is the weakest logic proved complete with respect to its reduced frames.[7] The axioms

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added to B to obtain C are just those needed for the completeness proofs for reduced frames to go through. The logic R is one of the main logics discussed by Anderson and Belnap [1], and it is one of the better known relevant logics.[8]

The propositional language under consideration will consist of a countably infinite set of atoms and compound formulas built from connectives from the set $\{\sim, \wedge, \vee, \rightarrow, \Box\}$. The rules and axioms for B are the following.[9]

A1 $A \rightarrow A$
A2 $A \wedge B \rightarrow A$, $A \wedge B \rightarrow B$
A3 $A \rightarrow A \vee B$, $B \rightarrow A \vee B$
A4 $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
A5 $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$
A6 $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$
A7 $\sim\sim A \rightarrow A$
A8 $\dfrac{A \rightarrow B \quad A}{B}$
A9 $\dfrac{A \quad B}{A \wedge B}$
A10 $\dfrac{A \rightarrow \sim B}{B \rightarrow \sim A}$
A11 $\dfrac{A \rightarrow B}{(C \rightarrow A) \rightarrow (C \rightarrow B)}$
A12 $\dfrac{A \rightarrow B}{(B \rightarrow C) \rightarrow (A \rightarrow C)}$

The logic C is obtained by adding to these axioms the following.

C1 $A \wedge (A \rightarrow B) \rightarrow B$
C2 $(A \rightarrow B) \rightarrow (\sim B \rightarrow \sim A)$
C3 $(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$

The logic R is obtained by adding to C the following axioms.

R1 $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
R2 $A \rightarrow ((A \rightarrow B) \rightarrow B)$

Let us now turn to the Routley-Meyer frame semantics for these logics.

Definition 1.1 (Routley-Meyer frames) A *Routley-Meyer frame* F is a quadruple $\langle K, N, R, ^* \rangle$, where $K \neq \emptyset$, $N \subseteq K$, R is a ternary relation on K, and * is a function from K to K such that the following conditions hold, where $a \leq b =_{Df} \exists x \in N\, Rxab$.

- $a \leq a$.
- If $a \leq b$ and $b \leq c$, then $a \leq c$.
- $a^{**} = a$.
- If $a \leq b$, then $b^* \leq a^*$.
- If $a \leq d$ and $Rdbc$, then $Rabc$.

A *modal Routley-Meyer frame* is a Routley-Meyer frame equipped with a binary relation S on K obeying the following conditions.

- If $a \leq b$ and Sbc, then Sac.

Note that because of the condition that $\leq$ is reflexive, N must be non-empty. For the rest of the paper, I will let context distinguish Routley-Meyer frames from the modal Routley-Meyer frames.

Definition 1.2 (Models) A Routley-Meyer model M is a frame F paired with a valuation V assigning pairs of atoms and points values from $\{0,1\}$ such that if $a \leq b$ and $V(p,a) = 1$, then $V(p,b) = 1$. Such a model is built on F.

A valuation V is extended to the whole language in the following way.

- $a \Vdash p$ iff $V(p,a) = 1$
- $a \Vdash \sim B$ iff $a^* \nVdash B$

- $a \Vdash B \wedge C$ iff $a \Vdash B$ and $a \Vdash C$
- $a \Vdash B \vee C$ iff $a \Vdash B$ or $a \Vdash C$
- $a \Vdash B \to C$ iff $\forall b, c(Rabc \wedge b \Vdash B \Rightarrow c \Vdash C)$
- $a \Vdash \Box B$ iff $\forall b(Sab \Rightarrow b \Vdash B)$

Definition 1.3 (Holding, validity) A formula A *holds in a model* M iff $\forall a \in N(a \Vdash A)$.
A formula A is valid in a frame F iff A holds in all models built on F.
A formula A is valid in a class of frames $\mathscr{C}$ iff A is valid in all frames $F \in \mathscr{C}$.

The logic B is sound and complete with respect to the class of all Routley-Meyer frames. Stronger logics can be obtained by imposing additional frame conditions. I will list the frame conditions corresponding to the additional axioms considered above. Additionally, $Rabcd$ is defined as $\exists x(Rabx \wedge Rxcd)$ and $Ra(bc)d$ is defined as $\exists x(Raxd \wedge Rbcx)$.

FC1 $Raaa$
FC2 $Rabc \Rightarrow Rac^*b^*$
FC3 $Rabcd \Rightarrow Rb(ac)d$
FR1 $Rabc \Rightarrow Rabbc$
FR2 $Rabc \Rightarrow Rbac$

The logic C is sound and complete with respect to the class of Routley-Meyer frames satisfying the conditions FC1–FC3, and R is sound and complete with respect to the class of frames satisfying, in addition, FR1 and FR2.

The modal extension L.M of a base relevant logic L is obtained by adding to the rules and axioms of L the following.[10]

- ($\Box\wedge$) $\Box A \wedge \Box B \to \Box(A \wedge B)$
- (RM) $\dfrac{A \to B}{\Box A \to \Box B}$

Additional modal relevant logics can be obtained by adding additional axioms and rules, and in this paper, the only additional rules and axioms explicitly considered will be the following.

- (K) $\Box(A \to B) \to (\Box A \to \Box B)$
- (T) $\Box A \to A$
- (4) $\Box A \to \Box\Box A$
- (5) $\sim\Box\sim A \to \Box\sim\Box\sim A$
- (Nec) $\dfrac{A}{\Box A}$

All of the modal relevant logics considered in this paper are contained in classical S5, under the translation τ defined as

- $\tau(p) = p$, for atoms,
- $\tau(\star B) = \star\tau(B)$, for $\star \in \{\Box, \sim\}$,
- $\tau(A \# B) = \tau(A) \# \tau(B)$, for $\# \in \{\wedge, \vee\}$, and
- $\tau(A \to B) = \sim\tau(A) \vee \tau(B)$.

It will be useful to have some terminology for various relationships to S5.

Definition 1.4 A formula A is S5-*contained* iff $\tau(A)$ is a theorem of S5.
A logic L is S5-*contained* iff for every theorem A of L, A is S5-contained.
A formula scheme is S5-*contained* iff for every instance A of the scheme, A is S5-contained.

A rule $\dfrac{A_1,\ldots,A_n}{B}$ is S5-*contained* iff whenever each of $A_1,\ldots,A_n$ are S5-contained, B is also S5-contained.

All the axioms above are S5-contained, the rules (RM) and (Nec) are S5-contained, so every logic specified axiomatically in this section is S5-contained.

The classical modal logic S5 plays the role in this paper of the strongest, plausible alethic modal logic. The intuition behind this is that a logic not contained in S5 is not a good candidate alethic modal logic, since it contains principles involving necessity that even the classical logician does not accept.[11] The frame conditions needed to validate these additional axioms and rules are not needed for this paper, so I omit them.

Definition 1.5 (Reduced frames) A frame F is *reduced* iff N is a singleton. For reduced frames, we use '0' for the single normal point. A frame is *unreduced* iff it is not reduced.

A model M is *reduced* iff it is built on a reduced frame.

Note that in a reduced model M, holding in M is equivalent to truth at 0. This becomes important below.

As some additional notation, I will use $FRM(\mathsf{L})$ for the class of Routley-Meyer frames on which L is valid and $FRM^r(\mathsf{L})$ for the subclass of reduced frames from $FRM(\mathsf{L})$. In this notation, C is sound and complete with respect to both $FRM(\mathsf{C})$ as well as $FRM^r(\mathsf{C})$, and R.M is sound and complete with respect to $FRM(\mathsf{R.M})$ but, as will be shown below, not with respect to $FRM^r(\mathsf{R.M})$.

That is sufficient background. I will now turn to Fuhrmann's incompleteness theorem.

2 Fuhrmann's theorem

The logic R.KT4 is obtained by adding (K), (T), (4), and (Nec) to R.M. Fuhrmann [12] shows that R.KT4 is incomplete with respect to the reduced frames for R.KT4. This is proved by providing a formula that is valid over the class of reduced frames for R.KT4 that is not a consequence of the axioms for R.KT4. The formula provided is

$$(\Box X) \qquad (\Box(A\to A)\to\Box A)\vee(\Box A\to\Box{\sim}(A\to A)).$$[12]

There are two things to note about this formula. First, as Furhmann observes, the $\Box$-free version,

$$(X) \qquad ((A\to A)\to A)\vee(A\to{\sim}(A\to A)),$$

is a theorem of R. It can be proved using many of the distinctive features of R. Given that excluded middle, $A\vee{\sim}A$, is a theorem of R, and the rule that if $\vdash_{\mathsf{R}} A\vee B$ and $\vdash_{\mathsf{R}} A\to C$, then $\vdash_{\mathsf{R}} C\vee B$, we can obtain it fairly readily. Using (R2), $A\to((A\to A)\to A)$ and the rule yields $\vdash_{\mathsf{R}} ((A\to A)\to A)\vee{\sim}A$. For the other disjunct, by (R2) and (C2), $\vdash_{\mathsf{R}} A\to({\sim}A\to{\sim}(A\to A))$, and by using the R-theorem, $(A\to(B\to C))\to(B\to(A\to C))$, $\vdash_{\mathsf{R}} {\sim}A\to(A\to{\sim}(A\to A))$. Using the given rule then yields the desired

$$\vdash_{\mathsf{R}} ((A\to A)\to A)\vee(A\to{\sim}(A\to A)).$$

The second thing to note is that $(\Box X)$ is a theorem of classical S4. In the classical setting, $(\Box X)$ is equivalent to a form of excluded middle, $\Box A\vee{\sim}\Box A$. This might give one the impression that the validity over reduced frames stays within the bounds of familiar classical modal logics. Such impressions are misleading, as will be shown in the next section.

The key to seeing that $(\Box X)$ is valid rests on what I will call Fuhrmann's lemma. The proof of Fuhrmann's lemma itself relies on the following lemma, which holds for all Routley-Meyer models in the present vocabulary, not just the reduced models.

Lemma 2.1 *For a Routley-Meyer model M, $A \to B$ holds in M iff for all $a \in K$, if $a \Vdash A$, then $a \Vdash B$.*

That much is standard and I will omit the proof.[13] Fuhrmann's lemma holds only for reduced models, where holding in a model amounts to truth at the normal world of that model. It is the analog in the case of reduced frames of the result showing that the rule (RM) is sound for unreduced frames.

Lemma 2.2 (Fuhrmann's lemma) *Let M be a reduced model. If $0 \Vdash A \to B$, then $0 \Vdash \Box A \to \Box B$.*

Proof Suppose $0 \Vdash A \to B$ holds in M. Suppose $a \Vdash \Box A$. Let b be an arbitrary point such that Sab. From the truth condition for $\Box$, $b \Vdash A$. By lemma 2.1, from the initial assumption it follows that for all $a \in K$, if $a \Vdash A$ then $a \Vdash B$. Therefore, $b \Vdash B$, which is sufficient to conclude that $a \Vdash \Box B$. Therefore, from lemma 2.1, $0 \Vdash \Box A \to \Box B$, as desired. □

The validity of $(\Box X)$ follows from Fuhrmann's lemma in combination with the validity of (X). The argument is essentially what I will use in the proof of lemma 3.2 below, so I will omit it here.

Fuhrmann shows that $(\Box X)$ is not a consequence of the axioms of R.KT4 by providing a six element matrix that invalidates it while validating R.KT4. The details of that argument are not needed here, so I will proceed to the more general incompleteness argument.

3 Incompleteness

In this section, I will show that any modal relevant logic extending C with a range of modal principles is incomplete with respect to its reduced frames. To show this, I will begin by showing that every instance of a certain formula scheme is a theorem of C and then I will show how this leads to trouble via Fuhrmann's lemma.

First, a lemma.

Lemma 3.1 *Each instance of the formula scheme $(B \to C) \vee {\sim}(B \to C)$ is a theorem of* C.

Proof Since C is complete with respect to $FRM(\mathsf{C})$, the argument will proceed via models.

Suppose that M is a countermodel. There is, then, a point $a \in N$ such that $a \nVdash (B \to C) \vee {\sim}(B \to C)$. So, $a \nVdash B \to C$ and $a \nVdash {\sim}(B \to C)$. By the former, there are points b, c such that $Rabc$, $b \Vdash B$, but $c \nVdash C$. From the latter, it follows that $a^* \Vdash B \to C$. By FC1, $Ra^*a^*a^*$, and by (FC2), Ra^*aa. This yields Ra^*abc, which by (FC3), yields $Ra(a^*b)c$. This unpacks to there being a d such that $Radc$ and Ra^*bd. The latter implies $d \Vdash C$. We also have $a \Vdash C \to C$, as $a \in N$, which with $Radc$ implies $c \Vdash C$, which is a contradiction.

There is, then, no countermodel to $(B \to C) \vee {\sim}(B \to C)$. By the completeness of C, $\vdash_{\mathsf{C}} (B \to C) \vee {\sim}(B \to C)$. □

The scheme shown valid in lemma 3.1 is not full excluded middle. It is, rather, excluded middle for implications, restricted excluded middle in the nomenclature of Anderson and Belnap [1], which is not often distinguished in discussions of relevant logics. While implicational formulas are frequently taken to be special or distinguished, for example by Anderson and Belnap [1] or Brady [7], as far as I know no one takes them to be special or distinguished in *that* way. Odd though it is, restricted excluded middle is enough for present purposes.

Lemma 3.2 *The formula scheme* $(\Box B \to \Box C) \vee {\sim}(B \to C)$ *is valid in* $FRM^r(\mathsf{C.M})$.

Proof By lemma 3.1, $(B \to C) \vee {\sim}(B \to C)$ is a theorem of C and so C.M. From the soundness of C.M, $(B \to C) \vee {\sim}(B \to C)$ is valid in $FRM^r(\mathsf{C.M})$. Let M be a model on such a frame from $FRM^r(\mathsf{C.M})$. Then $0 \Vdash (B \to C) \vee {\sim}(B \to C)$. So, either $0 \Vdash B \to C$ or $0 \Vdash {\sim}(B \to C)$. Applying Fuhrmann's lemma to $0 \Vdash B \to C$ yields $0 \Vdash \Box B \to \Box C$. So, we then have $0 \Vdash (\Box B \to \Box C) \vee {\sim}(B \to C)$. □

The formula scheme shown valid in lemma 3.2 is not S5-contained. This will be shown by invalidating an instance with an S5 countermodel.

Lemma 3.3 $(\Box p \to \Box q) \vee {\sim}(p \to q)$ *is not* S5*-contained.*

Proof To show the invalidity of $(\Box p \to \Box q) \vee {\sim}(p \to q)$ in classical S5, where the arrow is taken as the classical material conditional, take a Kripke frame with the universal accessibility relation and $W = \{x, y\}$. Set $x \Vdash p$, $y \Vdash p$, $x \Vdash q$, and $y \nVdash q$. We then have $x \Vdash \Box p$ but $x \nVdash \Box q$, falsifying the first disjunct at x. We also have $x \Vdash p \to q$, falsifying the second disjunct at x. □

Since C.M is S5-contained, it follows that $(\Box B \to \Box C) \vee {\sim}(B \to C)$ is not a theorem of C.M. Indeed, it is not a theorem of any logic L between C.M and classical S5. That is sufficient for the following incompleteness theorem.

Theorem 3.4 *No extension of* C.M *with* S5*-contained axioms is complete with respect to any subclass of* $FRM^r(\mathsf{C.M})$.

From the proofs of the preceding lemmas, one can obtain some further corollaries of this. Let L.5 be the extension of L.M with (K), (T), (4), (5), and (Nec). We can then sum the result up in the following corollary.

Corollary 3.5 *Let* L *be a logic between* C *and classical logic. Then no sublogic of* L.5 *is complete with respect to any subclass of* $FRM^r(\mathsf{L.5})$.

Further, no extension of C.M that adds axioms to the base logic, up to classical logic, or that adds S5-contained modal axioms or S5-contained rules, including the rule (Nec), will be complete with respect to the class of reduced frames validating the extension. This includes classical S5.

Corollary 3.6 *Let* L *be a logic between* C *and classical logic. Then no logic* L.Z *extending* L.M *with* S5*-contained axioms and* S5*-contained rules is complete with respect to any subclass of* $FRM^r(\mathsf{L.Z})$.

One final corollary captures an aspect of incompleteness with respect to reduced frames not present in the claims so far.

Corollary 3.7 *Let* L *be a logic between* B *and classical logic that has as a theorem* $A \vee {\sim}A$*, and let* L.Z *be an extension of* L.M *with* S5*-contained rules and axioms. Then* L.Z *is incomplete with respect to* FRM^r(L.Z).

While excluded middle is known to cause problems for completeness with respect to reduced frames for non-modal logics, a point to which I will return below, it is worth observing that excluded middle presents additional problems for completeness with respect to reduced modal frames.

One can, of course, add axioms to C.M that yield the troublesome validity. One simple example is adding $A \to \Box A$ and $\Box A \to A$, yielding a relevant version of the classical modal logic KT!, in which the box is redundant.[14] Another option is to add the offending disjunction, $(\Box B \to \Box C) \vee {\sim}(B \to C)$, as an axiom, although it is not clear that that will yield a completeness result. Further, this addition will result in an unattractive alethic modal logic, at least in the sense of being a logic that is not S5-contained. I will comment more on extension with rules below.

There are three things to observe about the results of this paper. The first is that the invalidity of $(\Box p \to \Box q) \vee {\sim}(p \to q)$ in C.M witnesses the failure of a disjunctive rule, namely the following.

$$\frac{A \vee (B \to C)}{A \vee (\Box B \to \Box C)}$$

Disjunctive rules are used in connection with reduced frames to obtain completeness results.[15] Seki [37] proves some completeness results for relevant logics extended with disjunctive modal rules, although those results are with respect to frames that may be unreduced. It is, as far as I know, an open question whether the reduced modal frames are well axiomatized by logics extended with disjunctive rules. As should be clear from the foregoing, the extension of any of the modal relevant logics extending C.M by disjunctive rules will be proper extensions, which extensions are not S5-contained.

The second thing to observe is that the incompleteness is here pinned on the validity of a disjunction, $(\Box B \to \Box C) \vee {\sim}(B \to C)$.[16] The proof that it is valid used conditions distinctive of the frames for C, and their modal extensions, to obtain a disjunctive theorem, which was used with Fuhrmann's lemma. For logics whose frames may lack some of those conditions, such as the logic TW and its sublogics, which lack (C1), a completeness result may be possible. As Slaney [39] showed, the theorems of TW and some its sublogics are, in a sense, built out of implications.[17] Seki [38] extended those results to many modal extensions of relevant logics. These logics have the disjunction property, meaning that if $A \vee B$ is a theorem of the logic, then either A is a theorem or B is.[18] That work suggests that the issues stemming from Fuhrmann's lemma may not arise in the context of reduced frames for modal relevant logics built on RW, TW, or weaker logics. Settling such questions will be left for future work.

As further evidence that disjunction is at fault, we note that for logics weaker than TW, similar issues arise for showing completeness with respect to reduced frames. As an example, take B extended with $A \vee {\sim}A$. It is the case that the formula $((q \to r) \to (p \to r)) \vee {\sim}(p \to r)$ is valid in the class of reduced frames for this logic. The proof is similar to that of lemma 3.1. For any model M on any frame in the class, $0 \Vdash (p \to q) \vee {\sim}(p \to q)$. Then, either $0 \Vdash p \to q$ or $0 \Vdash {\sim}(p \to q)$. As rule (A12) is sound for reduced models, we get $0 \Vdash (q \to r) \to (p \to r)$, so

$0 \Vdash ((q \to r) \to (p \to r)) \lor {\sim}(p \to q)$. This formula is not a theorem of B extended with $A \lor {\sim}A$.[19] Similarly, $B \lor {\sim}(A \land (A \to B))$ is valid in the class of all reduced frames for the extension of TW, or any of its sublogics containing B, with $A \lor {\sim}A$, which can be shown via similar reasoning.[20]

The third thing is a more philosophical point. There is a view suggested in work on reduced frames for relevant logics that the reduced frames are preferable, technically and philosophically, to the unreduced frames. This view is mentioned by Fuhrmann [12, 510] where he says, "It is sometimes thought to be intuitively more satisfactory, if the set [N] of 'distinguished worlds' could be reduced to a singleton set containing just the 'real world', 0." A similar thought can be found in the earlier cited work by Routley, Meyer, Slaney, and Giambrone. According to this view, the single normal world represents the real world, how things actually are or perhaps how they could be, which is thought to be a good thing.[21] One of the features of reduced frames that is highlighted as a virtue is that the rules for the base logic preserve truth at the single normal world, rather than merely preserving holding in a model, as in the unreduced models.[22] Additionally, in reduced models, the implications true at 0 line up neatly with the notion of implication in a model, a feature that need not hold for a given normal world in an unreduced model.[23]

The main result of this paper puts pressure on this view, as a general claim about frames for relevant logics, since the usual modal extensions of many relevant logics are simply incomplete with respect to their reduced frames. The logic of those reduced frames is, it turns out, not even S5-contained. A natural response, in keeping with the more recent work done by proponents of reduced models, such as Slaney and Brady, would be for the proponent of reduced frames to restrict claims about their adequacy to weaker logics, the logics for which completeness results for the modal extensions might hold.[24] Many of the proponents of reduced models argue for one or more of the contraction-free relevant logics, those with neither (C1) nor (R1).[25] While some may be inclined to take the result of this paper as a strike against reduced frames, one could take the results to be (yet another) a strike against the stronger relevant logics, assuming that completeness results are obtainable for the weaker logics.

I will close with a brief overview of the state of knowledge concerning frames for modal relevant logics. Many relevant logics are complete with respect to their reduced frames, in addition to their standard Routley-Meyer frame semantics. Many modal extensions of relevant logics are complete with respect to modal extensions of (generally, unreduced) Routley-Meyer frames. There are, however, some modal extensions of relevant logics that are incomplete, even with respect to the general frame semantics, as shown by Goble [14] and Mares [21]. Fuhrmann [12] showed that one modal relevant logic is incomplete with respect to its reduced frames. The contribution of this paper was to show that a wider range of modal relevant logics are incomplete with respect to their reduced frames. This result covers many modal extensions of the logics that were shown by Routley et al. [36] to be complete with respect to their reduced frame semantics. In order to obtain a completeness result for modal relevant logics with respect to reduced frames, one must look to the weaker logics.

Notes

1. See Routley et al. [36, Ch. 4], Slaney [40], and Giambrone [13].

2. Routley and Meyer [35]

3. The result was proved in Fuhrmann [11] but published as Fuhrmann [12]. I will cite the latter presentation here, with one exception.

   The logic shown complete in Routley and Meyer [35] is R.KT4, albeit with respect to non-standard reduced frames. The non-standardness is in the definition of $\leq$, which incorporates the binary, modal accessibility relation S in addition to the ternary relation R. Whereas the standard definition, in reduced frames, is $a \leq b$ iff $R0ab$, Routley and Meyer's definition is $a \leq b$ iff $\exists c(S0c \wedge Rcab)$. As Fuhrmann [12, 513] says, in reference to Routley and Meyer's reduced frames, "The semantics of [Routley and Meyer [35]] are thus ingeniousy tailor-made for R.KT4."

4. For overviews of the area, see Dunn and Restall [9] and Bimbó [3], and for overviews of some recent work, see Jago [18] and Bimbó [4]. For more detailed discussion, see Anderson and Belnap [1], Routley et al. [36], Anderson et al. [2], Read [28], Restall [31], Brady [6], and Mares [23].

5. The name C has been used, e.g. by Slaney [41], for a sublogic of R, now typically called RW, that is obtained by dropping (R1). That is a different logic from the logic C under discussion in this paper. It is worth noting that some relevant logics have 'C' added to their names to indicate extension by boolean negation, e.g. CE in the title of Mares [20].

6. In some early work on relevant logics, e.g. Meyer and Routley [24] and Fine [10], B includes excluded middle, $A \vee {\sim}A$, as an axiom, although this formulation appears to have become less common. Routley et al. [36, 289] calls that logic G, to distinguish it from their basic logic B. The B discussed in this paper does not include excluded middle as a theorem. I thank an anonymous referee for bringing this issue to my attention.

7. Slaney [40], with a correction by Giambrone [13], later showed how to extend completeness with respect to reduced frames to weaker logics.

8. See Mares [23] for an extended discussion of R.

9. Unary connectives bind more tightly than binary connectives, and conjunction and disjunction bind more tightly than the implication.

   The connective $\Diamond$ is being treated as defined, by ${\sim}\Box{\sim}$, throughout. One can take $\Diamond$ as primitive along with $\Box$, as Seki [37] does, or take $\Diamond$ as the sole modal primitive. The addition of $\Diamond$ requires some adjustments to the definition of a frame, as $\Diamond$ is interpreted on its own accessibility relation, and primitive $\Diamond$ need not be equivalent to ${\sim}\Box{\sim}$. Despite this, the results of this paper do not depend on the choice between these modal primitives.

10. The axiom ($\Box\wedge$) is sometimes called (C), e.g. by Fuhrmann [12]. The present name is adopted to avoid overloading 'C' any more.

11. This is not to say that S5 is being put forth as the correct alethic modal logic.

12. This is misprinted in Fuhrmann [12] but it is correct in Fuhrmann [11]. Furhmann's name for this formula is $(\Box X')$, but I drop the prime here.

13. It is worth observing that the definition of the heredity ordering, $\leq$, is important in the proof of lemma 2.1. One first shows that the preservation property stipulated for atoms in the definition of a model extends to all formulas, which argument relies on the definition of the ordering. With that in hand, one can prove lemma 2.1. The divergence between the usual definition of the heredity ordering and that used by Routley and Meyer [35], mentioned in footnote 3, comes to the fore at this point, since Routley and Meyer prove a version of lemma 2.1 that replaces $A \to B$ with $\Box(A \to B)$.

14. See Humberstone [17, 38].

15. See Brady [6, 7-9] for a brief overview.

16. Disjunction is known to lead to some problems for relevant logics dealing with necessity. In particular, disjunction is implicated in some issues arising for the relevant logic E, which is presented as the logic of relevance and necessity by Anderson and Belnap. Maksimova [19] showed that E diverges from R.KT4 over a formula involving disjunction, about which see also Anderson and Belnap [1, 351-352]. Further, Mares [20] shows that, unlike many other relevant logics, E is not conservatively extended by the addition of boolean negation, and the example of non-conservativeness involves disjunction.

    Disjunction causes some problems for non-modal relevant logics as well. Excluded middle is known to lead to problems with reduced modelings, as noted by Slaney [40], a point to which we return shortly, and disjunction presents some problems for providing an operational semantics, as in Urquhart [43], for the positive fragment of R, although Humberstone [15] showed how to solve those problems.

17. Slaney provides a sketch of proof for a normal form theorem showing that one can get a similar result for RW, but that depends on distinctive features of RW not enjoyed by all of its sublogics.

18. A logic L has the weaker property of being Halldén-complete iff if $A \vee B$ is a theorem of L and A and B do not share a propositional variable, then either A is a theorem of L or B is. Mares [22] examines Halldén-completeness in the context of modal relevant logics.

19. The following algebraic counterexample was found using John Slaney's program MaGIC, for more about which see http://users.cecs.anu.edu.au/ jks/magic.html. The set values is $\{0,1,2,3\}$, partially ordered as $0 \leq 1, 0 \leq 2, 1 \leq 3, 2 \leq 3$ with 3 as the sole designated value. Conjunction and disjunction are meet and join, respectively, on this lattice and implication and negation are interpreted using the following tabe.

$\to$	0	1	2	3	$\sim$
0	3	3	3	3	3
1	0	3	0	3	2
2	2	2	3	3	1
3	0	2	0	3	0

A counterexample valuation v has $v(p) = 2, v(q) = 0$, and $v(r) = 0$, which yields $v(((q \to r) \to (p \to r)) \vee {\sim}(p \to q)) = 1$. All axioms of B plus excluded middle get assigned the designated value, and the rules preserve the property of being designated. As $((q \to r) \to (p \to r)) \vee {\sim}(p \to q)$ is not designated on some valuation, it is not a theorem of the logic.

20. A slightly stronger version of this observation is mentioned by Slaney [40, 406]. The validity and resulting incompleteness hold in a logic extending TW with another axiom and rule in addition to excluded middle.

21. For discussion of this idea, see Standefer [42].

22. See, for example, Brady [5, 124]. It is not clear that the rules preserving truth at the base world is desirable for the modal rules, such (RM) or (Nec), which, in the setting of classical Kripke models, preserve truth in all worlds in a model rather than truth at a single world. See Humberstone [16] for a relevant discussion of the distinction between rules of inference and rules of proof.

23. This point is mentioned by Slaney [40, 395], for example.

24. One might include some work on the simplified semantics, Priest and Sylvan [27], Restall [30], and Restall and Roy [32], here as well, since those frames are all reduced. Note that the reduction condition is dropped in the presentation of Priest [26, Ch. 10]. I will also note that recently Brady [8] has become critical of the use of Routley-Meyer frames, reduced or not, although he still pushes for the use of a weak logic.

25. For more on these logics, see Restall [29]. Contraction axioms and rules lead to triviality in the context of naive theories of sets or of truth, about which, see Rogerson and Restall [34], Robles and Méndez [33], and Øgaard [25], among others, for more.

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