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Identity in Mares-Goldblatt Models for Quantified Relevant Logic

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Abstract

Mares and Goldblatt (*The Journal of Symbolic Logic*, *71*(01), 163–187, 2006) provided an alternative frame semantics for two quantified extensions of the relevant logic R. In this paper, I show how to extend the Mares-Goldblatt frames to accommodate identity. Simpler frames are provided for two zero-order logics en route to the full logic in order to clarify what is needed for identity and substitution, as opposed to quantification. I close with a comparison of this work with the Fine-Mares models for relevant logics with identity and a discussion of constant and variable domains.

Keywords Relevant logic · Mares-Goldblatt frames · Identity

One of the motivating intuitions behind relevant logics is that necessary truths, even logical truths, need not be implied by all formulas.[1] This intuition is meant to extend beyond propositional logics to include identity. After all, if $p \rightarrow (p \rightarrow p)$ is not valid, why should $p \rightarrow (t = t)$ be? There are plausible axiom systems for relevant logics with identity that do not have $p \rightarrow (t = t)$ as a theorem. There is then a question of *models* for such axiom systems. Such models would have to avoid validating obvious irrelevancies, such as $p \rightarrow (t = t)$, while maintaining some of the logical behavior of identity, such as the substitution of identicals.

One of the distinctive applications of identity in relevant logic is Dunn's *relevant predication*.[2] There is an intuitive difference between saying of Mount Taranaki that it is in New Zealand and saying of Mount Fuji that it is such that Mount Taranaki is

[1] For good overviews of relevant logics, see [13] and [3]. For overviews of recent work in the field see [24] and [4]. For more in depth treatments, see [1, 2, 38, 41], and [30].
[2] For more on relevant predication, see Dunn [9–11] and Kremer [25, 26].

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in New Zealand. The predicate “is such that Mount Taranaki is in New Zealand” does not really involve its object, as opposed to “is in New Zealand”, which is a paradigm of an object involving predicate. Dunn defines relevant predication as

$$(\rho x A(x))a =_{Df} \forall x(x = a \rightarrow A(x)).$$

Relevantly predicating $A(x)$ of a is saying that being identical to a implies having the property $A(x)$. True relevant predications establish more of a connection between their subject and predicate than standard predications. While Mount Fuji may be such that Mount Taranaki is in New Zealand, something being Mount Fuji intuitively falls short of implying the location of Mount Taranaki.

Of importance for the theory of relevant predication is the fact that there are different formulations of the substitution axiom for identity, a conjunctive form,

$$A(s) \,\&\, s = t \rightarrow A(t),$$

and an iterated conditional form,

$$A(s) \rightarrow (s = t \rightarrow A(t)).$$

The iterated conditional form, it turns out, does not sit well with the relevance intuitions, as argued by Dunn [9]. Hence, I will focus on the conjunctive form here. Mares [29] shows that allowing substitution into the scope of a conditional in the conjunctive substitution axiom has serious consequences for the theory of relevant predication, and so he considers a restricted form of the conjunctive substitution axiom.[3] Kremer [27] discusses motivations for different axiomatizations of relevant identity. As is apparent from this work, there is much to explore regarding identity in relevant logics. Mares [32] explores connections between the relevant biconditional and identity. Øgaard [35] looks at substitution of identicals in relevant logics.

There has been some work on models for quantified relevant logics with and without identity. The pioneering work of Mares [29] provided models for a range of relevant logics with identity and quantifiers, building on the work of Fine [15, 19]. The Fine models for quantified relevant logics, and consequently the Fine-Mares models for relevant identity, have not found widespread adoption within the relevant logic community.[4] Brady [5, 6] provided a kind of algebraic semantics for a weak relevant logic with quantifiers. There has not been an extension of Brady’s semantics to deal with identity.

Priest [37, ch. 24] develops constant domain models for quantified relevant logics, as well as the extension with identity. Priest uses the Tarskian truth condition that the truth of every instance suffices for the truth of the universal. Priest provides a tableau system that is sound and complete for his models, rather than a Hilbert-style axiom system like the other authors mentioned above.

[3] The restricted and unrestricted forms of the substitution axiom are, respectively, (I3) and (I4) below.

[4] See Brady [7] for some criticisms of Fine’s models, and see Logan [28] for discussion and defense of Fine’s models. See Goldblatt [20] for a conservative extension result using Fine’s models.

Mares and Goldblatt [33] provided alternative models for quantified relevant logics, focussing on a pair of logics extending R, and Goldblatt and Kane [22] used similar techniques to provide models for a range of relevant and substructral logics with propositional quantification.[5] Mares [31] provides a philosophical interpretation of the framework. Mares and Goldblatt equip standard Routley-Meyer models with a set of admissible propositions, which are used to provide a *non-Tarskian* truth condition for the universal quantifier. The truth condition is non-Tarskian in the sense that the satisfaction by all assignments need not suffice for the truth of the universal.

In this paper I will show how to obtain models for logics of identity extending the relevant logic R.[6] I will begin by providing some formal background on R and its frame semantics. The logics of identity differ on the strength of their substitution axioms, following the pattern of [29]. I will then show how to enrich Routley-Meyer models for R to interpret identity, using ideas from the Fine-Mares models. I will then incorporate some ideas from [33] to provide models for a strengthened logic of identity. Following this, I will briefly sketch the additions needed for quantifiers, as developed by Mares and Goldblatt and show how to incorporate identity into the models for quantification.[7] Finally, I will close by discussing some salient differences between the models for identity presented here and the Fine-Mares models.

1 Background

The formulas are built up as follows, where Var is a set of variables, Con a set of constants, and Terms the union of Var and Con.[8]

$$A ::= Ft_1, \ldots, t_n \mid \mathsf{t} \mid {\sim}A \mid (A \,\&\, B) \mid (A \to B)$$

The set of atoms will be denoted by At. I will take $A \leftrightarrow B$ to be defined as $(A \to B) \,\&\, (B \to A)$, $A \vee B$ as ${\sim}({\sim}A \,\&\, {\sim}B)$, $A \circ B$ as ${\sim}(A \to {\sim}B)$, and $\exists x A$ as ${\sim}\forall x{\sim}A$. To cut down on parentheses, negation will bind tightest, followed by conjunction and disjunction.

The logic under consideration will be in the framework FMLA, i.e. a set of formulas, the theorems.[9] Initially, the logic of interest will be the logic R, to whose zero-order fragment I will add identity axioms to obtain the logics $\mathsf{R}^=$ and $\mathsf{R}^=_{sub}$. Following that, quantifiers will be added to the language and the quantified relevant logic of interest will be QR, which is the focus of [33]. The logic R has the following

[5] Goldblatt [21] develops admissible proposition frames for (classical) quantified modal logics. Chapter 6 of that work presents *cover semantics* as another approach to the semantics of quantified R.

[6] The restriction to R is to connect with the work of [33] on quantified relevant logics. The axioms and frame and model conditions for identity do not seem to rely on proprietary features of R, but it is beyond the scope of this paper to explore identity in the context of other logics.

[7] Ferenz [14] has developed a different approach to incorporating identity into Mares-Goldblatt models.

[8] Var and Con are assumed to be disjoint, both countable, and Var infinite.

[9] Humberstone [23, 103ff.]

axioms and rules, where '$\Rightarrow$' in the rules is used to mark off the premises from the conclusion in the rules.

(A1)	$A \to A$	(A10)	$(A \to (B \to C)) \to (B \to (A \to C))$
(A2)	$A \,\&\, B \to A$, $A \,\&\, B \to B$	(A11)	$\sim\sim A \to A$
(A3)	$(A \to B) \,\&\, (A \to C) \to (A \to B \,\&\, C)$	(A12)	$(A \to \sim B) \to (B \to \sim A)$
(A4)	$A \to A \vee B$, $B \to A \vee B$	(A13)	t
(A5)	$(A \to C) \,\&\, (B \to C) \to (A \vee B \to C)$	(A14)	$A \leftrightarrow (\mathsf{t} \to A)$
(A6)	$A \,\&\, (B \vee C) \to (A \,\&\, B) \vee (A \,\&\, C)$	(A15)	$(A \to (B \to C)) \leftrightarrow ((A \circ B) \to C)$
(A7)	$(A \to B) \to ((B \to C) \to (A \to C))$		
(A8)	$(A \to B) \to ((C \to A) \to (C \to B))$	(R1)	$A, A \to B \Rightarrow B$
(A9)	$(A \to (A \to B)) \to (A \to B)$	(R2)	$A, B \Rightarrow A \,\&\, B$

While this section is concerned just with the logic R, there are a few notions, such as the notion of a theorem, that will be used in the context of R as well as its extensions below. A *proof* of a formula A in a logic L is a sequence of formulas, each of which is either an axiom of L or follows from earlier members of the sequence by a rule of L. A formula A is a *theorem* of a logic L, $\vdash_{\mathsf{L}} A$, iff there is a proof of A in the logic L. Let us now turn to the basic frames.

I will begin with the basic frames for the zero-order logic of R before enriching these frames to interpret identity in later sections. I will use the following definitions for the heredity ordering and the compositions of the ternary relation.

– $a \leq b =_{Df} \exists x \in O \; Rxab$
– $Rabcd =_{Df} \exists x (Rabx \,\&\, Rxcd)$
– $Ra(bc)d =_{Df} \exists x (Raxd \,\&\, Rbcx)$

All Routley-Meyer frames considered in this paper will be frames for the logic R. I will define the frames in steps, beginning with frames for the propositional logic and then proceeding to the zero-order frames.

Definition 1 (Routley-Meyer frame) A Routley-Meyer frame, F is a quadruple $\langle K, O, ^*, R\rangle$, where $K \neq \emptyset$, $O \subseteq K$, $^* : K \mapsto K$ such that $a^{**} = a$, and $R \subseteq K^3$, satisfying the following conditions.

– $a \in O \,\&\, a \leq b \Rightarrow b \in O$.
– $\leq$ is reflexive, transitive, and antisymmetric.
– $a \leq b \Rightarrow b^* \leq a^*$.
– $d \leq a \,\&\, Rabc \Rightarrow Rdbc$.
– $Rabc \Rightarrow Rbac$.
– $Rabcd \Rightarrow Ra(bc)d$.
– $Raaa$.
– $Rabc \Rightarrow Rac^*b^*$.

Definition 2 (Zero-order Routley-Meyer frame) A zero-order Routley-Meyer frame G is a quintuple $\langle K, O, ^*, R, I\rangle$, where $\langle K, O, ^*, R\rangle$ is a Routley-Meyer frame and the domain $I \neq \emptyset$.

Definition 3 (Zero-order Routley-Meyer model) A zero-order Routley-Meyer model M is a pair $\langle F, V\rangle$, where F is a zero-order Routley-Meyer frame and V is an interpretation function such that

- $V(P) : I^n \mapsto \wp(K)$,
- $V(c) \in I$, for $c \in \mathsf{Con}$, and
- for $a, b \in K$, if $a \leq b$ and $a \in V(P)(i_1, \ldots, i_n)$, then $b \in V(P)(i_1, \ldots, i_n)$.

A model M is *built on* a frame F iff M is $\langle F, V\rangle$, for some V.

An assignment $f : \omega \mapsto I$ is a function from the variables to the domain, also written I^ω below.

As suggested by the definition of assignment, there will, following Mares and Goldblatt, be a canonical ordering on the countable set of variables, which permits us to slide between them and natural numbers.

Following Mares and Goldblatt, for $t \in \mathsf{Terms}$, define $Vf(t)$ as follows.

$$Vf(t) = \begin{cases} V(t) & t \in \mathsf{Con} \\ f(t) & t \in \mathsf{Var} \end{cases}$$

The valuation V can be extended to a satisfaction relation, $\Vdash_V$, according to the following clauses.

- $f, a \Vdash_V Pt_1, \ldots, t_n$ iff $a \in V(P)(Vf(t_1), \ldots, Vf(t_n))$
- $f, a \Vdash_V \mathsf{t}$ iff $a \in O$
- $f, a \Vdash_V A \,\&\, B$ iff $f, a \Vdash_V A$ and $f, a \Vdash_V B$
- $f, a \Vdash_V A \to B$ iff $\forall bc(Rabc \,\&\, f, b \Vdash_V A \Rightarrow f, c \Vdash_V B)$
- $f, a \Vdash_V {\sim}A$ iff $f, a^* \nVdash_V A$

We can define the *truth set* for A on V and f as $|A|_{Vf} = \{a \in K : f, a \Vdash_V A\}$. Each formula defines a function $|A|_V : I^\omega \mapsto \wp(K)$, defined as $|A|_V(f) = |A|_{Vf}$. It will be useful to write $V(P)(t_1, \ldots, t_n)(f)$ for $V(P)(Vf(t_1), \ldots, Vf(t_n))$, especially when changes of assignment are being considered more. Once propositional functions are considered below, the former notation will be more substantive, as a notation for propositional functions. I will put $a \Vdash A$ iff $f, a \Vdash_V A$ for all f.

A formula A is *satisfied in a model M* by assignment f iff for all $a \in O$, $f, a \Vdash_V A$. A formula A *holds in a model M* iff for all $a \in O$, $a \Vdash A$. A formula A is *valid in a frame F* iff A holds in all models M built on F. A formula A is *valid in a class* $\mathfrak{F}$ of frames iff A is valid in F for each $F \in \mathfrak{F}$. A closed formula A is *valid*, $\models_{\mathsf{R}} A$ iff it is vaild in the class of all zero-order Routley-Meyer frames.

For a formula A, say that assignments f and g agree on the free variables of A iff for all n, if x_n is free in A, then $f(n) = g(n)$. I will record a fact here that is clear from the definition of satisfaction.

Lemma 1 (Locality Lemma) *If f and g agree on the free variables of A, then* $|A|_{Vf} = |A|_{Vg}$.

At this point in the paper, there are no bound variables, but this lemma remains true once quantifiers, and bound variables, are added.

A version of the Heredity Lemma holds for our models.

Lemma 2 (Heredity Lemma) *If $a \leq b$ and $f, a \Vdash_V A$, then $f, b \Vdash_V A$*

Proof The proof is a straightforward induction on the complexity of A. □

Say that a set $X \subseteq K$ is *hereditary* iff for all $a, b \in K$, if $a \in X$ and $a \leq b$, then $b \in X$. It then follows that for all valuations V, all assignments f, and all formulas A, $|A|_{Vf}$ is hereditary.

From the Heredity Lemma we can prove the Verification Lemma.

Lemma 3 (Verification Lemma) *A formula $A \to B$ is satisfied in a model M by f iff for all $a \in K$, $f, a \Vdash_V A$ only if $f, a \Vdash_V B$.*

The Verification Lemma is used extensively in proofs of Soundness. The cases of the Soundness proof will not be presented here, as they would repeat material readily available.

Theorem 1 (Soundness) *If $\vdash_{\mathsf{R}} A$, then $\models_R A$*

Proof The proof is by a straightforward induction on the length of the proof of A. □

The proofs of Completeness in this paper use the techniques of Henkin-style canonical model construction. As is usual, we need some definitions.

Definition 4 Let Γ and Δ be sets of formulas. Say that the pair (Γ, Δ) is L-inconsistent iff there are $A_1, \ldots, A_n \in \Gamma$ and $B_1, \ldots, B_m \in \Delta$ such that $\vdash_{\mathsf{L}} (A_1 \mathbin{\&} \cdots \mathbin{\&} A_n) \to (B_1 \vee \cdots \vee B_m)$.

The pair (Γ, Δ) is L-consistent iff it is not L-inconsistent.

Definition 5 (Theories) A set of formulas Γ is a L-*theory* iff for every formula B, if (Γ, B) is L-inconsistent, then $B \in \Gamma$.

An L-theory Γ is *prime* iff $A \vee B \in \Gamma$ implies $A \in \Gamma$ or $B \in \Gamma$.

A theory Γ is L-*regular* iff $\vdash_{\mathsf{L}} A$ implies $A \in \Gamma$.

There are a couple of things to remark on in these definitions. First, these definitions are more general than is needed for this section, as we will reuse them in later sections. In this section, the logic L is just R, while in later sections, it can be extensions of R. Second, the definition of L-theory is equivalent to another common definition used in the context of relevant and substructural logics.[10] The main point of these definitions is the Prime Extension Lemma.

Lemma 4 (Prime Extension Lemma) *If* (Γ, Δ) *is* L*-consistent, then there is a prime* L*-theory* $\Sigma \supseteq \Gamma$ *such that* (Σ, Δ) *is* L*-consistent.*

Proof See [39, 92-95]. □

Next, we define the canonical model for the logic R. The 'R-' prefix will be omitted, in what follows, to aid readability. Following [33], we will assume there are countably many distinct constants in Con.

- K is the set of prime theories.
- O is the set of regular, prime theories.
- For $a, b, c \in K$, $Rabc$ iff $\{A \circ B : A \in a \,\&\, B \in b\} \subseteq c$.
- $a^* = \{A : \sim A \notin a\}$.
- $I = \mathsf{Con}$.
- For all $c \in \mathsf{Con}$, $V(c) = c$.
- For n-ary predicate letters P, $V(P)(c_1, \ldots, c_n) = \{a \in K : Pc_1, \ldots, c_n \in a\}$.

The canonical frame, so defined, satisfies all the conditions for being a zero-order Routley-Meyer frame, and the canonical valuation satisfies the conditions for being a valuation. Next, for all $f \in I^\omega$, and A^f is the closed formula

$$A[x_0/f(0), x_1/f(1), \ldots, x_n/f(n), \ldots],$$

the result of carrying out the substitution of $f(n)$ for x_n, for all free variables of A.

Lemma 5 *For all formulas* A, $A^f \in a$ *iff* $f, a \Vdash_V A$.
For all closed formulas A, $A \in a$ *iff* $a \Vdash A$.

Theorem 2 (Completeness) *For closed formulas* A, *if* $\models_{\mathsf{R}} A$, *then* $\vdash_{\mathsf{R}} A$.

Proof Assume that it is not the case that $\vdash_{\mathsf{R}} A$. Then the pair $(\mathsf{R}, \{A\})$ is R-consistent, where the first member of the pair is the set of theorems of R. In the canonical model, there is a regular, prime theory $a \in O$ such that $A \notin a$, so $a \nVdash A$. Therefore it is not the case that $\models_{\mathsf{R}} A$. □

[10]The other common definition of an L-theory is a set of formulas X such that both (i) if $A \in X$ and $B \in X$, then $A \,\&\, B \in X$, and (ii) if $A \in X$ and $\vdash_{\mathsf{L}} A \to B$, then $B \in X$. These two definitions coincide for a wide range of relevant logics. For condition (i), if $A \in \Gamma$ and $B \in \Gamma$, then (Γ, A) and (Γ, B) are L-inconsistent. It follows that $\vdash_{\mathsf{L}} (C \to A) \,\&\, (C \to B)$, for some conjunction C from Γ. Then from axiom (A3) and (R1), $\vdash_{\mathsf{L}} C \to A \,\&\, B$, so $(\Gamma, A \,\&\, B)$ is L-inconsistent. Thus, $A \,\&\, B \in \Gamma$. For (ii), suppose $A \in \Gamma$ and $\vdash_{\mathsf{L}} A \to B$. It follows that (Γ, A) is L-inconsistent, so there is a conjunction C from Γ such that $\vdash_{\mathsf{L}} C \to A$. If $\vdash_{\mathsf{L}} C \to A$ and $\vdash_{\mathsf{L}} A \to B$ jointly suffice for $\vdash_{\mathsf{L}} C \to B$, as they do in any relevant logic extending B, then (Γ, B) is L-inconsistent. Therefore, $B \in \Gamma$, as desired.

The other direction of the equivalence is straightforward. Suppose X is an L-theory as defined in the previous paragraph. Suppose (X, A) is L-inconsistent. Then there is a conjunction C of formulas from X such that $\vdash_{\mathsf{L}} C \to A$. Since all the conjuncts of C are in X, $C \in X$, by condition (i). From condition (ii), $A \in X$, as desired.

With the background for the relevant logic R and basic Routley-Meyer frames in place, we can now turn to the primary topic of the paper, identity.

2 Zero-Order Identity

In this section, we will augment zero-order Routley-Meyer frames to obtain frames and models appropriate for interpreting identity. We extend the language with a binary predicate for identity, =. There are a few constraints on how this predicate can be interpreted, which will be familiar to the reader of [29]. The interpretation needs to be reflexive on the domain, at least over O, to deliver $t = t$, but it need not be everywhere reflexive, else $p \to t = t$ will be valid. It needs to be symmetric and transitive. There will need to be constraints on how identity holding at a point relates to identity at the point's star and how identities constrain valuations to ensure substitution is sound.

The way that the constraints are formally met is to add to each frame a family of indexed relations, $\approx_a$, for $a \in K$, where $\approx_a \subseteq I \times I$. The indexed relations are then required to satisfy several frame and model conditions corresponding to the constraints. As we will see, these conditions are appropriate for identity that licenses substitution of identicals in $\to$-free formulas.

First, say that a point a is *weakly symmetric* in i, j iff if $Vf(u) = i$ and $Vf(v) = j$, then for each predicate letter P, $a \in V(Pt_1, \ldots, t_n)(f)$ iff $a \in V(Ps_1, \ldots, s_n)(f)$, where $t_\ell = s_\ell$ if $s_\ell \notin \{u, v\}$, and where $t_\ell = u$ or $t_\ell = v$ if $s_\ell \in \{u, v\}$.

Definition 6 (Identity frames) A zero-order identity frame F is sextuple $\langle K, O, ^*, R, I, \{\approx_a\}_{a \in K}\rangle$, where the initial quintuple is a zero-order Routley-Meyer frame and the family of relations obeys the following conditions.

(ID1) $a \leq b \Rightarrow \approx_a \subseteq \approx_b$.
(ID2) If $a \in O$, then $(i, i) \in \approx_a$.
(ID3) $(i, j) \in \approx_a \Rightarrow (j, i) \in \approx_a$.
(ID4) $(i, j) \in \approx_a \,\&\, (j, k) \in \approx_a \Rightarrow (i, k) \in \approx_a$.
(ID5) $(i, j) \in \approx_a \,\&\, (j, k) \in \approx_{a^*} \Rightarrow (i, k) \in \approx_{a^*}$.

To define a model, we need to coordinate the family of relations and the valuation. For this, we define the notion of confomity.

Definition 7 (Conformity) A valuation V *conforms* to a Routley-Meyer frame with identity iff for all $f \in I^\omega$, $s, t \in$ Terms, $a \in K$,

$$a \in V(=)(Vf(s), Vf(t)) \text{ iff } (Vf(s), Vf(t)) \in \approx_a.$$

Definition 8 (Identity models) A zero-order identity model M is a pair $\langle F, V\rangle$, where F is a zero-order identity frame, V is a valuation conforming to F and obeying the conditions from the zero-order models, as well as the following two conditions.

(ID6) If $(i, j) \in \approx_a$, then a^* is weakly symmetric in i, j.
(ID7) If $Vf(u) = i$, $Vf(v) = j$, $a \in V(Pt_1, \ldots, t_n)(f)$, and $(i, j) \in \approx_a$, then $a \in V(Ps_1, \ldots, s_n)$, where $s_\ell = t_\ell$, if $t_\ell \notin \{u, v\}$, and $s_\ell \in \{u, v\}$, if $t_\ell \in \{u, v\}$.

Conditions (ID1)–(ID4) are fairly straightforward. Conditions (ID5) and (ID6) capture a maximality intuition about the Routley star. According to this intuition, a^* "asserts what [the point a] does not deny."[11] Formally, they yield the soundness of substitution in the scope of negation, as one would expect. Finally, (ID7) is the condition on valuations ensuring substitution of identicals for atomic formulas.

From this section forward, I will mostly be concerned with validity with respect to a class of models, rather than with respect to a class of frames. A formula A holds in a model M iff for all $a \in O$, $a \Vdash A$. A formula is valid in a class of models iff it holds in all models in that class. A formula A is valid, $\models_{\mathsf{R}^=} A$, iff A is valid in the class of all zero-order identity models.

Using the definition of conformity, we can obtain the following derived satisfaction condition for identity.

– $f, a \Vdash_V s = t$ iff $(Vf(s), Vf(t)) \in \approx_a$

To see this, note that $f, a \Vdash_V s = t$ iff $a \in V(=)(Vf(s), Vf(t))$, which by conformity, is equivalent to $(Vf(s), Vf(t)) \in \approx_a$.

The Heredity Lemma extends to the zero-order identity models.

Lemma 6 *If $a \leq b$ and $f, a \Vdash_V A$, then $f, b \Vdash_V A$.*

Proof The new case is when A is $s = t$. This is covered by condition (ID1). □

To axiomatize the logic, we will add the following axioms to R to obtain the logic $\mathsf{R}^=$.

(I1) $t = t$
(I2) $s = t \to t = s$
(I3) $A \,\&\, s = t \to A'$, where A is $\to$-free, and A' is the result of substituting one or more occurrences of s for t in A.

Note that $s = t \,\&\, t = u \to s = u$ follows from the axioms for $\mathsf{R}^=$, as can be seen from [29, 12]. I will say that a formula A is a *theorem* of $\mathsf{R}^=$, $\vdash_{\mathsf{R}^=} A$ iff there is a proof of A from the axioms and rules of $\mathsf{R}^=$.

Theorem 3 (Soundness) *If $\vdash_{\mathsf{R}^=} A$, then $\models_{\mathsf{R}^=} A$.*

Proof The proof is by induction on the length of the proof of A. The new cases are those involving (I1)–(I3).

For (I1), by (ID2), for $a \in O$, $(i, i) \in \approx_a$. For any $t \in$ Terms, $(Vf(t), Vf(t)) \in \approx_a$, so $f, a \Vdash_V t = t$. Therefore, $\models_{\mathsf{R}^=} t = t$.

[11] See [12, 332].

For (I2), suppose $f, a \Vdash_V s = t$. So, $(Vf(s), Vf(t)) \in \approx_a$. By (ID3), $(Vf(t), Vf(s)) \in \approx_a$, so $f, a \Vdash_V t = s$. Therefore, $s = t \to t = s$ holds in M on f, by the Verification Lemma.

For (I3), we proceed by induction on the $\to$-free formula A. We will suppose that $Vf(s) = i$ and $Vf(t) = j$.

Subcase: A has the form $Pt_1, \ldots, t_n$. Assume $f, a \Vdash_V Pt_1, \ldots, t_n$ and $f, a \Vdash_V s = t$, so $a \in V(Pt_1, \ldots, t_n)(f)$ and $(i, j) \in \approx_a$. By (ID7), $a \in V(Ps_1, \ldots, s_n)(f)$, where $s_\ell = t_\ell$, if $t_\ell \notin \{s, t\}$ and otherwise s_ℓ may be either of s or t. So, $f, a \Vdash_V A'$, as desired.

Subcase: $t = u$. Assume $f, a \Vdash_V t = u$ and $f, a \Vdash_V s = t$. Then $(Vf(t), Vf(u)) \in \approx_a$ and $(Vf(s), Vf(t)) \in \approx_a$. By (ID4), $(Vf(s), Vf(u)) \in \approx_a$, so $f, a \Vdash_V s = u$, as desired. The other possibilities for this subcase are similar.

Subcase: $\sim Pt_1, \ldots, t_n$. Assume $f, a \Vdash_V \sim Pt_1, \ldots, t_n$ and $f, a \Vdash_V s = t$, so $f, a^* \nVdash_V Pt_1, \ldots, t_n$. So, $a^* \notin V(Pt_1, \ldots, t_n)(f)$. Suppose $f, a \nVdash_V \sim Ps_1, \ldots, s_n$, where $s_\ell = t_\ell$, if $t_\ell \notin \{s, t\}$ and otherwise s_ℓ may be either of s or t. So, $a^* \in V(Ps_1, \ldots, s_n)(f)$. Since $(i, j) \in \approx_a$, a^* is weakly symmetric in i, j by (ID6), $a^* \in V(Pt_1, \ldots, t_n)(f)$, and so $f, a \nVdash_V \sim Pt_1, \ldots, t_n$, contradicting the assumption. Therefore, $f, a \Vdash_V \sim Ps_1, \ldots, s_n$, which is A', as desired.

Subcase: $\sim(t = u)$. Assume $f, a \Vdash_V \sim(t = u)$ and $f, a \Vdash_V s = t$. Then $(i, j) \in \approx_a$ and $f, a^* \nVdash_V t = u$, so $(Vf(t), Vf(u)) \notin \approx_{a^*}$. Suppose $f, a \nVdash_V \sim(s = u)$. Then $f, a^* \Vdash_V s = u$. So, $(i, Vf(u)) \in \approx_{a^*}$, which together with $(j, i) \in \approx_a$, from (ID3), yields $(j, Vf(u)) \in \approx_{a^*}$ from (ID5). But then $f, a^* \Vdash_V t = u$, and so $f, a \nVdash_V \sim(t = u)$, contradicting the assumption. Therefore, $f, a \Vdash_V \sim(s = u)$. As with the identity subcase, the other possibilities are similar.

Subcases: $\sim\sim B$, $B \,\&\, C$, $\sim(B \,\&\, C)$. These are handled by the inductive hypothesis.

Since the desired conclusion is obtained in all subcases, I conclude that axiom (I3) is valid. □

To prove Completeness, we adapt the canonical model construction of Section 1. The definitions for the canonical model are carried over, using $\mathsf{R}^=$ rather than R, with the following additional definition.

- $(s, t) \in \approx_a$ iff $s = t \in a$.

It remains to define the family of relations and show that the canonical model obeys conditions (ID1)–(ID7).

Lemma 7 *The canonical model for* $\mathsf{R}^=$ *obeys the frame conditions* (ID1)–(ID5).

Proof For (ID1), suppose $a \leq b$ and $(s, t) \in \approx_a$. It follows that $s = t \in a$ and there is a $c \in O$ such that $Rcab$. So $s = t \in b$, and so $(s, t) \in \approx_b$.

For (ID2), suppose $a \in O$. As $\vdash_{\mathsf{R}^=} s = s$, $s = s \in a$, by definition, so $(s, s) \in \approx_a$.

For (ID3), suppose $(s, t) \in \approx_a$. Then $s = t \in a$. From axiom (I2), $t = s \in a$, so $(t, s) \in \approx_a$.

For (ID4), note that $\vdash_{\mathsf{R}^=} s = t \,\&\, t = u \to s = u$. Suppose $(s, t) \in \approx_a$ and $(t, u) \in \approx_a$. Then $s = t \,\&\, t = u \in a$, whence $s = u \in a$ so $(s, u) \in \approx_a$.

For (ID5), suppose $(s, t) \in \approx_a$ and $(t, u) \in \approx_{a^*}$. Suppose that $(s, u) \notin \approx_{a^*}$. Then $s = u \notin a^*$, so $\sim(s = u) \in a$. As $s = t \in a$, by (I3), $\sim(t = u) \in a$. But then, $t = u \notin a^*$, so $(t, u) \notin \approx_{a^*}$, contradicting the assumption. Thus, $(s, u) \in \approx_{a^*}$. □

Lemma 8 *The canonical model for* R$^=$ *obeys the model conditions* (ID6)–(ID7) *and conforms to the canonical frame.*

Proof For (ID6), suppose $Vf(u) = s$, $Vf(v) = t$, $(s, t) \in \approx_a$ and $a^* \in V(Pt_1, \ldots, t_n)(f)$. Suppose that $a^* \notin V(Ps_1, \ldots, s_n)(f)$, where the s_ℓ's and t_ℓ's satisfy the conditions for weak symmetry in s, t. So, $\sim(Ps_1, \ldots, s_n)^f \in a$. By assumption, $Vf(u) = s$, $Vf(v) = t$, and $(s, t) \in \approx_a$, so $u = v \in a$. By (I3), we then have $(\sim Pt_1, \ldots, t_n)^f \in a$, so $(Pt_1, \ldots, t_n)^f \notin a^*$, contradicting the assumption. Therefore, $a^* \in V(Ps_1, \ldots, s_n)(f)$.

For the other direction in weak symmetry, suppose $Vf(u) = s$, $Vf(v) = t$, $(s, t) \in \approx_a$, $a^* \notin V(Pt_1, \ldots, t_n)(f)$, and $a^* \in V(Ps_1, \ldots, s_n)(f)$, where the s_ℓ's and t_ℓ's satisfy the conditions for weak symmetry in s, t. So, $(\sim Pt_1, \ldots, t_n)^f \in a$. By assumption, $Vf(u) = s$, $Vf(v) = t$, and $(s, t) \in \approx_a$, so $(u = v)^f \in a$. By (I3), we then have $(\sim Ps_1, \ldots, s_n)^f \in a$, so $(Ps_1, \ldots, s_n)^f \notin a^*$, contradicting the assumption. Therefore, $a^* \notin V(Ps_1, \ldots, s_n)(f)$.

For (ID7), suppose $Vf(u) = s$, $Vf(v) = t$, $(s, t) \in \approx_a$ and $a \in V(Pt_1, \ldots, t_n)(f)$. Suppose that $a \notin V(Ps_1, \ldots, s_n)(f)$, where the s_ℓ's and t_ℓ's satisfy the conditions of (ID7). Then $(Pt_1, \ldots, t_n)^f \in a$ and $(u = v)^f \in a$, but $(Ps_1, \ldots, s_n)^f \notin a$. As $(u = v \,\&\, Pt_1, \ldots, t_n)^f \in a$, by (I3), $(Ps_1, \ldots, s_n)^f \in a$, which is a contradiction. Therefore, $a \in V(Ps_1, \ldots, s_n)(f)$, as desired.

For conformity, note that $a \in V(=)(Vf(s), Vf(t))$ iff $(s = t)^f \in a$, which is equivalent to $(Vf(s), Vf(t)) \in \approx_a$. □

Lemma 9 (Satisfaction Lemma) *For all formulas A, for all assignemnts f, $A^f \in a$ iff $f, a \Vdash_V A$.*

Proof The proof is by induction on the structure of A. The non-identity atomic case and the connective cases are all as usual, so we will do the identity case.

$$\begin{aligned} (s = t)^f \in a \quad &\text{iff} \quad s^f = t^f \in a \\ &\text{iff} \quad (Vf(s), Vf(t)) \in \approx_a \\ &\text{iff} \quad a \in V(=)(Vf(s), Vf(t)) \\ &\text{iff} \quad f, a \Vdash_V s = t \end{aligned}$$

□

Theorem 4 (Completeness) *For closed formulas A, if $\models_{\mathsf{R}^=} A$, then $\vdash_{\mathsf{R}^=} A$.*

Proof Suppose that is is not the case that $\vdash_{\mathsf{R}^=} A$. Construct the canonical model for R$^=$. As A is not a theorem, there is a regular, prime theory a such that $A \notin a$. From the satisfaction lemma, we then have $a \nVdash A$, as A is closed. Therefore, it is not the case that $\models_{\mathsf{R}^=} A$. □

The axioms for R$^=$ are sound and complete with respect to the zero-order identity models. The substitution axiom, (I3), has a restriction, which one may want to drop.

One can validate the unrestricted axiom, provided an additional model condition is adopted. Stating that condition will require the addition of some of the machinery used by Mares and Goldblatt in their frames for quantified relevant logic, so we will turn to that now.

3 Mares-Goldblatt Proposition Frames

Mares and Goldbatt enrich zero-order Routley-Meyer frames with sets of admissible propositions and propositional functions in order to interpret the quantifiers. Propositions can do some work even when quantifiers are not in the language, and I will use them to provide the model conditions for a strengthened substitution axiom, which drops the restriction of (I3) that A be $\rightarrow$-free:

(I4) $A \,\&\, s = t \rightarrow A'$, where A' is the result of substituting one or more occurrences of s for t in A.

Let $\mathsf{R}^{=}_{sub}$ be the set of theorems resulting from adding all instances of (I4) to the axioms of $\mathsf{R}^{=}$.

In the context of Routley-Meyer frames, propositions are hereditary sets of points, i.e. if $a \in X$ and $a \leq b$, then $b \in X$. This is in contrast to Kripke frames for classical modal logic, in which propositions can be arbitrary sets of points. The set of propositions in a frame will have to obey some closure conditions. For a given frame, we define $-$ and $\Rrightarrow$ as operations on $\wp(K)$, for all $X, Y \subseteq K$.

- $-X = \{a \in K : a^* \notin X\}$
- $X \Rrightarrow Y = \{a \in K : \forall b \in K \forall c \in K((Rabc \,\&\, b \in X) \Rightarrow c \in Y)\}$

Note that if X and Y are hereditary sets on a frame, then so are $-X$, $X \Rrightarrow Y$, $X \cup Y$, and $X \cap Y$.

A propositional function on a frame is a function $\phi : I^{\omega} \mapsto \mathsf{Prop}$, mapping assignments to propositions. Following Mares and Goldblatt, I will use 'ϕ' and 'ψ' for propositional functions. Operations on propositional functions will be defined pointwise as in Table 1.

Now we can define propositional Routley-Meyer frames.

Definition 9 A *propositional Routley-Meyer frame* is an octuple,

$$\langle K, O, ^*, R, I, \{\approx_a\}_{a \in K}, \mathsf{Prop}, \mathsf{PropFun}\rangle,$$

where the first six components comprise a zero-order identity Routley-Meyer frame, Prop is a non-empty set of hereditary subsets of K, and PropFun is a non-empty set

Table 1 Pointwise operations on propositional functions

$(\phi \cap \psi)f$	=	$\phi f \cap \psi f$
$(\phi \Rrightarrow \psi)f$	=	$\phi f \Rrightarrow \psi f$
$(-\phi)f$	=	$-(\phi f)$

propositional functions from assignments to Prop, satisfying the following closure conditions.

CProp $O \in$ Prop, and if $X, Y \in$ Prop, then $X \cap Y \in$ Prop, $X \Rrightarrow Y \in$ Prop, and $-X \in$ Prop.
CTee $\phi_O \in$ PropFun, where $\phi_O(f) = O$, for all $f \in I^\omega$.
CImp If $\phi, \psi \in$ PropFun, then $\phi \Rrightarrow \psi \in$ PropFun.
CConj If $\phi, \psi \in$ PropFun, then $\phi \cap \psi \in$ PropFun.
CNeg If $\phi \in$ PropFun, then $-\phi \in$ PropFun.

Looking forward to the extension with quantifiers, to define models from propositional Routley-Meyer frames, we will need the valuations V to assign atomic formulas propositional functions. The propositional function assigned to an atomic formula by a valuation V is the defined as $V(Pt_1, \ldots, t_n)(f) = V(P)(Vf(t_1), \ldots, Vf(t_n))$, making good on the comment that the former notation would receive a more substantive interpretation. We will say that a valuation V is *admissible* iff $V(Pt_1, \ldots, t_n) \in$ PropFun, for all atomic formulas. When concerned with identity, as we are here, one also needs to coordinate the relations $\approx_a$ and PropFun, but that is guaranteed by the conformity condition.

Definition 10 A *propositional Routley-Meyer model M* is a pair $\langle F, V\rangle$, where F is a propositional Routley-Meyer frame and V is an admissible valuation that conforms to F

The definition of a model ensures that the truth sets of atomic formulas are all hereditary. In fact, in a given model, $|A|_{Vf} \in$ Prop, for any formula A and there is $\phi_A \in$ PropFun such that $|A|_{Vf} = \phi_A(f)$.

Lemma 10 *Let A be an arbitrary formula. In any propositional Routley-Meyer model, $|A|_V \in$ PropFun. So, $|A|_{Vf} \in$ Prop.*

Proof The proof is similar to that of corollary 4.1 in [33]. The atomic case for the Ackermann constant is handled by CTee and the atomic cases for identities and non-identity atomic formulas are handled by the definition of model. The cases for complex formulas are handled by CConj, CNeg, and CImp. □

The Heredity Lemma follows from the fact that the closure conditions on Prop and PropFun preserve the property of being hereditary. The conformity condition properly coordinates the satisfaction condition with the propositional function for identity as well, as demonstrated by the following argument.

$$\begin{aligned}(Vf(s), Vf(t)) \in \approx_a \text{ iff } & a \in V(=)(Vf(s), Vf(t)) \\ \text{iff } & a \in V(s = t)(f)\end{aligned}$$

The first line is the conformity condition, and the second equivalence is justified by the definition of propositional function.

The point of using Prop and PropFun at this stage is to provide a frame condition that yields the soundness of (I4). For this, we need further definitions and notation.

Definition 11 (Conflation) For $f, g \in I^{\omega}$, $f \sim^{i}_{j} g$ iff for all $n \in \omega$,

- $f(n) = g(n)$, or
- $f(n) \neq g(n)$ and either $f(n) = i$ and $g(n) = j$ or $f(n) = j$ and $g(n) = i$.

For a given frame, for $a \in K$, say a is (i, j)-conflating iff for all $\phi \in \mathsf{PropFun}$, for all $f, g \in I^{\omega}$, $(f \sim^{i}_{j} g \Rightarrow (a \in \phi(f) \Leftrightarrow a \in \phi(g)))$.[12]

We can then define a *strong propositional Routley-Meyer frame* as a propositional Routley-Meyer frame that additionally satisfies the condition

CFullSub if $(i, j) \in \approx_a$ then a is (i, j)-conflating.

A little more scaffolding is needed for Soundness.

Definition 12 Let $B[x_1/t_1, \ldots, x_n/t_n]$ be the result of simultaneous substitution of the term t_i for x_i in B, provided t_i is free for x_i.

A *skeleton* of a formula A is a formula B, whose unbound terms are only free variables, $x_1, \ldots, x_n$, each of which occurs only once, such that there are terms $t_1, \ldots, t_n$ such that $A = B[x_1/t_1, \ldots, x_n/t_n]$.[13]

For $f \in I^{\omega}$ and $j \in I$, define $f[j/n]$ as

$$f[j/n] = \langle f0, f1, \ldots f(n-1), j, f(n+1), \ldots \rangle$$

and $f[j_1/n_1, \ldots, j_{m+1}/n_{m+1}]$ as

$$f[j_1/n_1, \ldots, j_{m+1}/n_{m+1}] = (f[j_1/n_1, \ldots, j_m/n_m])[j_{m+1}/n_{m+1}],$$

where the n_i are distinct

At this point, no terms are bound and all terms are free for all others. This phrasing was chosen to accommodate the addition of quantifiers and variable binding. We will record a fact about skeletons and syntactic substitution.

Lemma 11 *If A' is the result of substituting zero or more occurrences of term s for a term t in A, provided s is free for t in A, then there is a formula B that is a skeleton for both A and A'*

Next, we record a fact about skeletons and satisfaction.

Lemma 12 *Suppose B is a skeleton for A, where $B[x_{n_1}/t_1, \ldots, x_{n_m}/t_m] = A$. Suppose that $Vf(t_i) = j_i$, for each i such that $1 \leq i \leq m$. Then*

$$f[j_1/n_1, \ldots, j_m/n_m], a \Vdash_V B \textit{ iff } f, a \Vdash_V A.$$

[12] This terminology is based on that of Ripley [40], although the logical features are different. I thank Lloyd Humberstone for suggesting the term “conflation” in this context.

[13] Since function symbols are not in the language, this definition coincides with that of a *matrix* from Priest [36, 17], minus the stipulation that the variables are increasing order. Since I will not deal with function symbols here, I will not further address the possible divergence between the two notions. Thanks to Dave Ripley for pointing out Priest’s term to me.

Corollary 1 *Suppose B is a skeleton for A, where $B[x_{n_1}/t_1, \ldots, x_{n_m}/t_m] = A$. Suppose that $Vf(t_i) = j_i$, for each i such that $1 \leq i \leq m$. Then*

$$\phi_B(f[j_1/n_1, \ldots, j_m/n_m]) = \phi_A(f).$$

Let us turn to Soundness.

Theorem 5 *If $\vdash_{\mathsf{R}^{=}_{sub}} A$ then $\models_{\mathsf{R}^{=}_{sub}} A$.*

Proof Most of the cases are handled as before, although there is a new case, (I4), which subsumes (I3).

Suppose that $f, a \Vdash_V A \,\&\, s = t$. So, $f, a \Vdash_V A$ and $f, a \Vdash_V s = t$. Suppose $Vf(s) = i$ and $Vf(t) = j$, so it follows that $(i, j) \in \approx_a$. By the condition CFullSub, a is (i, j)-conflating. Suppose that $f, a \nVdash_V A'$. By Lemma 11, there is a formula B that is a skeleton for both A and A'. To keep notation simple, suppose that A and A' differ on exactly one term occurrence, so $A = B[x_{n_1}/t_1, \ldots, x_{n_m}/t_m, x_{n_{m+1}}/t]$ and $A' = B[x_{n_1}/t_1, \ldots, x_{n_m}/t_m, x_{n_{m+1}}/s]$.

Let g be the assignment $f[j_1/n_1, \ldots, j_m/n_m]$, where $Vf(t_i) = j_i$, for $1 \leq i \leq m$. As $f, a \Vdash_V A$, by Lemma 12, $g[i/n_{m+1}], a \Vdash_V B$. By Lemma 12, $g[j/n_{m+1}], a \Vdash_V B$ iff $f, a \Vdash_V A'$. So, $a \notin \phi_B(g[j/n_{m+1}])$ while $a \in \phi_B(g[i/n_{m+1}])$. Further, we have $g[i/n_{m+1}] \sim^i_j g[j/n_{m+1}]$, by definition. But, this contradicts the fact that a is (i, j)-conflating. Therefore, $f, a \Vdash_V A'$, as desired. □

We now turn to Completeness.

4 Completeness for Zero-Order Propositional Models

The proof of Completeness uses the techniques of [33]. We carry over the canonical model construction of the previous section, so that the set K of points is the set of prime $\mathsf{R}^{=}_{sub}$-theories, the set O is the subset of $\mathsf{R}^{=}_{sub}$-regular theories, and so on. We make the following additions to incorporate propositions and propositional functions. Note that the valuation clause or atomic formulas has been replaced.

- For closed formulas A, $||A|| = \{a \in K : A \in a\}$.
- Prop $= \{||A|| : A$ is a closed formula$\}$.
- For each formula A, $\phi_A : I^\omega \mapsto$ Prop is defined as $\phi_A(f) = ||A^f||$.
- PropFun $= \{\phi_A : A$ is a formula$\}$.
- For n-ary predicate letters P, $V(P)(c_1, \ldots, c_n) = ||Pc_1, \ldots, c_n||$.

For much of the proof, we can use the work of Mares and Goldblatt, as well as the arguments of the previous section. The canonical frame satisfies conditions (ID1)–(ID7). It remains to be verified that the condition CFullSub holds.

To show that the canonical model obeys CFullSub, it will be useful to prove a small lemma on substitution.

Definition 13 ((s, t)-variants) Suppose that two formulas A and B have a common skeleton C, $A = C[x_1/t_1, \ldots, x_n/t_n]$, and $B = C[x_1/s_1, \ldots, x_n/s_n]$. A and B are

(s, t)*-variants* iff for all i such that $1 \leq i \leq n$, either $t_i = s_i$, or $t_i \neq s_i$ but both $t_i \in \{s, t\}$ and $s_i \in \{s, t\}$.

Lemma 13 *Suppose that A and B are* (s, t)*-variants. Then,* $\vdash_{\mathsf{R}^{=}_{sub}} A \,\&\, s = t \to B$.

Proof Suppose that A and B are (s, t)-variants. Let C be a common skeleton such that $A = C[x_1/t_1, \ldots, x_n/t_n]$, and $B = C[x_1/s_1, \ldots, x_n/s_n]$. Let $m_1, \ldots, m_\ell$ be the subscripts such that for $1 \leq p \leq \ell$ $t_{m_p} \neq s_{m_p}$ and $t_{m_p} = t$. Let $D = C[x_1/t'_1, \ldots, x_n/t'_n]$, where $t'_r = s$, if r is one of the m_p's, and $t'_r = t_r$ otherwise. Let $E = C[x_1/t''_1, \ldots, x_n/t''_n]$, where $t''_r = r$, if r is one of the m'_p's, and $t''_r = t'_r$ otherwise.

By (ID4), $\vdash_{\mathsf{R}^{=}_{sub}} A \,\&\, s = t \to D$, as D is A with some occurrences of s replacing some occurrences t. By (ID4), $\vdash_{\mathsf{R}^{=}_{sub}} D \,\&\, s = t \to E$, as E is D with some occurrences of t replacing some occurrences s. From (A2), (R2), and (A3), it follows that $\vdash_{\mathsf{R}^{=}_{sub}} A \,\&\, s = t \to D \,\&\, s = t$, whence $\vdash_{\mathsf{R}^{=}_{sub}} A \,\&\, s = t \to E$, by (A7) and (R1).

Finally, note that E is B, so $\vdash_{\mathsf{R}^{=}_{sub}} A \,\&\, s = t \to B$, as desired. □

Lemma 14 *The canonical model for* $\mathsf{R}^{=}_{sub}$ *obeys condition* CFullSub.

Proof Suppose $(i, j) \in \approx_a$. Then $i = j \in a$. Suppose a is not (i, j)-conflating, so there is a $\phi \in$ PropFun such that for some $f, g \in I^\omega$ such that $f \sim^i_j g$, either

- $a \in \phi(f)$ and $a \notin \phi(g)$, or
- $a \notin \phi(f)$ and $a \in \phi(g)$.

Suppose $a \in \phi(f)$ and $a \notin \phi(g)$. From the definition of PropFun, there is a formula A such that $\phi = \phi_A$. Without loss of generality, we can assume that A is a skeleton of some formula. Then A^f and A^g differ in so far as the latter has zero or more occurrences of i where the former has j and zero or more occurrences of j where the former has i. Then A^f and A^g are (i, j)-variants. By the previous lemma, $\vdash_{\mathsf{R}^{=}_{sub}} A^f \,\&\, i = j \to A^g$. Since $a \in \phi_A(f)$, $a \in ||A^f||$, so $A^f \in a$. So, $A^f \,\&\, i = j \in a$. We then have $A^g \in a$. This implies $a \in ||A^g||$, which implies $a \in \phi_A(g)$, contradicting the assumption.

The argument for the other disjunct is similar. Therefore, we conclude the canonical model obeys CFullSub. □

That completes the condition check. Since the other conditions are verified by the work of Mares and Goldblatt, we conclude that the canonical model is, in fact, a $\mathsf{R}^{=}_{sub}$-model. Next, we need a lemma showing membership is truth.

Lemma 15 *For all formulas A, and all assignments f,* $|A|_{Vf} = \phi_A(f)$.

Proof For the base case, both the identity and non-identity atomic formulas are taken care of by the definition of the model. The inductive cases are all handled by the argument of lemma 9.6 of [33]. □

With that, we can claim Completeness.

Theorem 6 *For closed formulas A, if* $\models_{\mathsf{R}^{=}_{sub}} A$, *then* $\vdash_{\mathsf{R}^{=}_{sub}} A$.

Let us take stock of what has been done so far. We have two zero-order logics with identity, $\mathsf{R}^{=}$ and $\mathsf{R}^{=}_{sub}$. These logics differ over the strength of the substitution axiom, the restricted (I3) versus the unrestricted (I4), respectively. Models for these two logics have been defined, and soundness and completeness have been shown. We now add quantifiers to the language and consider properly first-order logics of identity.

5 Quantified Relevant Logic

The quantified relevant logic of primary interest here is QR, as that is the focus of [33]. It is obtained by adding the following axioms and rule to R.

(Q1) $\forall x A \to A[x/t]$, where x is free for t in A

(R3) $A \to B \Rightarrow A \to \forall x B$, where x is not free in A

As noted by [33, 182], the extensional confinement axiom,

$$\text{(EC)} \qquad \forall x(A \vee B) \to A \vee \forall x B,$$

where x is not free in A, is not a theorem of QR. Adding it yields the stronger logic RQ, which logic will not be considered further here. There are two first-order logics of identity that will be considered, $\mathsf{QR}^{=}$ and $\mathsf{QR}^{=}_{sub}$. The former is obtained by adding (I1), (I2), and (I3) to QR, and the latter adds (I4) to $\mathsf{QR}^{=}$.

I will begin with the frames for QR. These will be propositional Routley-Meyer frames without the family of binary relations and its attendant conditions but with some additional closure operations on Prop and PropFun. First, there is one further operation needed to handle quantification, defined by [33]. Define $\sqcap : \wp(\wp(K)) \mapsto \wp(K)$ for $S \subseteq \wp(K)$, by

$$\bigsqcap S = \bigcup\{X \in \mathsf{Prop} : X \subseteq \bigcap S\}.$$

We then define new operations on propositional functions for each $n \in \omega$

$$(\forall_n \phi) f = \bigsqcap_{j \in I} \phi(f[j/n]).$$

Definition 14 A QR-frame F is a septuple $\langle K, O, R, ^*, I, \mathsf{Prop}, \mathsf{PropFun}\rangle$, where the first five components comprise a zero-order Routley-Meyer frame, Prop is a non-empty set of hereditary subsets of K, and PropFun is a non-empty set of functions in $I^{\omega} \mapsto \mathsf{Prop}$ obeying the following conditions.

- CProp $O \in \mathsf{Prop}$, and if $X, Y \in \mathsf{Prop}$, then $X \cap Y \in \mathsf{Prop}$, $X \Rrightarrow Y \in \mathsf{Prop}$, and $-X \in \mathsf{Prop}$.
- CTee $\phi_O \in \mathsf{PropFun}$, where $\phi_O(f) = O$, for all $f \in I^{\omega}$.
- CImp If $\phi, \psi \in \mathsf{PropFun}$, then $\phi \Rrightarrow \psi \in \mathsf{PropFun}$.
- CConj If $\phi, \psi \in \mathsf{PropFun}$, then $\phi \cap \psi \in \mathsf{PropFun}$.

CNeg If $\phi \in$ PropFun, then $-\phi \in$ PropFun.
CAll If $\phi \in$ PropFun, then $\forall_n \phi \in$ PropFun, for each $n \in \omega$.

A QR-model M is pair $\langle F, V \rangle$ where F is a QR-frame and V is an admissible valuation.

All but the last of these conditions on QR-frames were used in the definition of propositional Routley-Meyer frames. The condition CAll is required to get a frame for the quantificational logic QR as defined by Mares and Goldblatt.

Next, define the set of x_n-variants of f as $x_n v f = \{f[j/n] \in I^\omega : j \in I\}$. The valuation V can be extended to a satisfaction relation, $\Vdash_V$, by using the clauses from the previous sections with the addition of the following clause for the universal quantifier.

– $f, x \Vdash_V \forall x A$ iff there is $X \in$ Prop such that $a \in X$ and $X \subseteq \bigcap_{g \in xvf} |A|_{Vg}$.

The definitions of a formula A being satisfied in a model by an assignment, holding in a model, being valid on a frame, being valid in a class of frames, and being valid, in symbols $\models_{\mathsf{QR}} A$, are adapted in a straightforward way to the present context of QR-frames.

Theorem 7 (Soundness and completeness) *For all formulas A, $\vdash_{\mathsf{QR}} A$ iff $\models_{\mathsf{QR}} A$.*

Proof For proofs, see [33]. □

With that background in place, I will turn to the addition of identity to quantification and models for $\mathsf{QR}^=$.

6 Models for Identity and Quantification

In this section, I will present models for $\mathsf{QR}^=$ and prove Soundness and Completeness. Following this, I will briefly discuss the adaptations needed to provide models for $\mathsf{QR}^=_{sub}$.

Definition 15 A $\mathsf{QR}^=$-frame F is a propositional Routley-Meyer frame obeying the following additional condition.

CAll If $\phi \in$ PropFun, then $\forall_n \phi \in$ PropFun, for each $n \in \omega$.

A $\mathsf{QR}^=$-model M is pair $\langle F, V \rangle$ where F is a QR-frame and V is an admissible valuation that conforms to F and obeys the following additional conditions.

CUnivSub If $a \in (\phi_{s=t} \cap \phi_{\forall x_n A})(f)$, then there is $X \in$ Prop such that $a \in X$ and $X \subseteq \bigcap_{g \in x_n v f} \phi_{A'}(g)$, where A is $\to$-free and A' is the result of substituting one or more occurrences of s for t in A.

CNegUnivSub If $a \in \phi_{s=t}(f)$ and $a^* \in \phi_{\forall x_n A'}(f)$, then there is $X \in$ Prop such that $a^* \in X$ and $X \subseteq \bigcap_{g \in x_n v f} \phi_A(g)$, where A is $\to$-free and A' is the result of substituting one or more occurrences of s for t in A.

As before, the definitions of a formula A being satisfied in a model by an assignment, holding in a model, being valid in a class of models, and being valid, $\models_{\mathsf{QR}^=} A$, are adapted to the present context in the straightforward way.

As should be clear from the definitions, a $\mathsf{QR}^=$-frame F can be viewed as either a propositional Routley-Meyer frame satisfying the additional condition CAll or as a QR-frame augmented with a family of binary relations $\{\approx_a\}_{a \in K}$ on $I \times I$ obeying conditions (ID1)–(ID5). The two conditions on models, CUnivSub and CNegUnivSub, are used in the proof of Soundness for $\mathsf{QR}^=$, along with the conditions of conformity, (ID6), and (ID7). Following the proof of Soundness below, I will comment on them further.

Next, I will prove soundness for $\mathsf{QR}^=$ with respect to the class of all $\mathsf{QR}^=$-models.

Theorem 8 (Soundness) *If* $\vdash_{\mathsf{QR}^=} A$, *then* $\models_{\mathsf{QR}^=} A$.

Proof Mares and Goldblatt [33] show soundness for QR. It remains to show that axioms (I1)–(I3) are valid. The proofs that axioms (I1) and (I2) are valid proceed as in Section 2. There are two new subcases for the induction for (I3), namely when A is of the form $\forall x B$ and when A is of the form $\sim\forall x B$.

Subcase: $\forall x B$. Suppose that $f, a \Vdash_V \forall x B$ and $f, a \Vdash_V s = t$. We can suppose that $x \notin \{s, t\}$, as then the substitution would be vacuous and $f, a \Vdash_V \forall x B'$, as $B' = B$. By Lemma 10, $a \in (\phi_{s=t} \cap \phi_{\forall x B})(f)$. By CUnivSub, there is $X \in$ Prop such that $a \in X$ and $X \subseteq \bigcap_{g \in x v f} |B'|_{Vg}$. This means that $f, a \Vdash_V \forall x B'$, as desired.

Subcase: $\sim\forall x B$. Suppose that $f, a \Vdash_V \sim\forall x B$ and $f, a \Vdash_V s = t$. We can suppose that $x \notin \{s, t\}$, as then the substitution would be vacuous and $f, a \Vdash_V \sim\forall x B'$. Assume $f, a \nVdash_V \sim\forall x B'$. Then $f, a^* \Vdash_V \forall x B'$. By Lemma 10, $a \in \phi_{s=t}(f)$ and $a^* \in \phi_{\forall x B'}(f)$. By CNegUnivSub, there is $X \in$ Prop such that $a^* \in X$ and $X \subseteq \bigcap_{g \in x v f} |B|_{Vg}$. But then, $f, a^* \Vdash_V \forall x B$, whence $f, a \nVdash_V \sim\forall x B$, contradicting the assumption. Therefore, $f, a \Vdash_V \sim\forall x B'$, as desired. □

As can be seen from the proof, the conditions CUnivSub and CNegUnivSub were key to the universal and negated universal subcases for the soundness of axiom (I3).[14]

[14]The requirement of the conditions CUnivSub and CNegUnivSub for the soundness of (I3) suggests a further restricted form of substitution that permits substitution only into the scope of formulas built out of negation, conjunction, and disjunction, in particular excluding the universal quantifier. This division is suggested by the models, as the universal quantifier is the only connective that appeals to Prop, but it does not seem like an antecedently natural line to draw. This division might be more natural if one thinks of the universal quantifier less like a generalized conjunction and more like a kind of necessity operator, something that brings with it a degree of opacity.

It is worth dwelling on these conditions. Without them, one might try to argue, by induction, that, for a given a, if $f, a \Vdash_V s = t$, then $f, a \Vdash_V A$ iff $f, a \Vdash_V A'$. In the universal quantifier case for $\forall x B$, one would be able to conclude, that for each $g \in xvf$, $a \in |B|_{Vg}$ iff $a \in |B'|_{Vg}$. But this would hold only for the given a, falling short of $|B|_{Vg} = |B'|_{Vg}$, which would suffice for the desired conclusion.[15] Instead, one would only obtain the weaker

$$a \in \bigcap_{g \in xvf} |B|_{Vg} \text{ iff } a \in \bigcap_{g \in xvf} |B'|_{Vg},$$

with no guarantee that there is a set $X \in \mathsf{Prop}$ with $a \in X$ and $X \subseteq \bigcap_{g \in xvf} |B'|_{Vg}$, as $\bigcap_{g \in xvf} |B'|_{Vg}$ may fail to be in Prop.[16]

A bit of reflection on the models and the truth condition for the universal quantifier reveals the issue. In the attempted argument given in the previous paragraph, one can establish only that each x-variant of B holds at the point a. However, as Mares and Goldblatt point out, the Tarskian truth condition, that a universally quantified formula is true when every instance is true, is not sufficient, in general, for the truth of the universally quantified formula. The attempted argument only delivers each instance, satisfying the Tarskian truth condition but falling short of the Mares-Goldblatt condition for the truth of the universal.

The additional conditions, CUnivSub and CNegUnivSub, are needed to guarantee that there are the requisite propositions to witness the truth of the universal quantifier and negated universal quantifier. The set Prop of propositions is, in a sense, a *global* feature of the model, in contrast to the *local* features reachable via the heredity ordering; a proposition may contain points a and b that are incomparable in the heredity ordering. While there may be points such that $s = t$ holds at them, the truth sets of $\forall x Rxs$ and $\forall x Rxt$, in a model and assignment, can be largely disjoint. This is similar to how the truth sets of their respective instances, as given by the range of x-varying assignments, need not have much in common. It is for these reasons that the existence of a proposition witnessing the truth of the former universal formula is no guarantee that there will be a witnessing proposition for the latter. The case in which one has a negated universal just compounds the issues, as the identity need not hold at the star point under consideration.

The proof for Completeness is fairly straightforward, given what has gone before. The canonical model construction does not need any new definitions from those of

[15] If, for each $g \in xvf$, $|B|_{Vg} = |B'|_{Vg}$, then $\bigcap_{g \in xvf} |B|_{Vg} = \bigcap_{g \in xvf} |B'|_{Vg}$. From the assumption of the case, there is an $X \in \mathsf{Prop}$ with $a \in X$ such that $X \subseteq \bigcap_{g \in xvf} |B|_{Vg}$. The latter identity then ensures that the same X works for $\bigcap_{g \in xvf} |B'|_{Vg}$.

[16] As shown by lemma 4.5 of [33, 171], there are cases in which $\bigcap_{g \in xvf} |B'|_{Vg}$ is guaranteed to be in Prop, namely when one of I and Prop is finite, or when Prop is full, i.e. contains all hereditary subsets of K.

the previous sections. For the verification that the frame satisfies the various conditions, we can reproduce the arguments of earlier sections. The main addition is verifying that the canonical model satisfies the conditions CUnivSub and CNegUnivSub. Before proving that, we will prove a lemma about universal propositions and their instances.

Lemma 16 *In the canonical model for* QR$^{=}$, $||(\forall x_n A)^f|| \subseteq \bigcap_{g \in x_n v f} \phi_A(g)$.

Proof First, observe that $x_n v f = \{g \in I^\omega : \exists c \in I(g = f[c/n])\}$.

From (Q1), $||(\forall x_n A)^f|| \subseteq ||A[x_n/c]^f||$, for all $c \in I$. But, $A[x_n/c]^f = A^{f[c/n]}$, so by the observation

$$||(\forall x_n A)^f|| \subseteq \bigcap_{g \in x_n v f} ||A^g||,$$

whence

$$||(\forall x_n A)^f|| \subseteq \bigcap_{g \in x_n v f} \phi_A(g).$$

□

Lemma 17 *The canonical model for* QR$^{=}$ *satisfies the conditions* CUnivSub *and* CNegUnivSub.

Proof For CUnivSub, suppose $a \in (\phi_{s=t} \cap \phi_{\forall x_n A})(f)$, for $\rightarrow$-free A. We need to show that there is $X \in$ Prop such that $a \in X$ and $X \subseteq \bigcap_{g \in x_n v f} \phi_{A'}(g)$. Since $a \in (\phi_{s=t} \cap \phi_{\forall x_n A})(f)$, $a \in ||(s = t \,\&\, \forall x_n A)^f||$, so $(s = t \,\&\, \forall x_n A)^f \in a$. From (I3), it follows that $(\forall x_n A')^f \in a$. By Lemma 16,

$$||(\forall x_n A')^f|| \subseteq \bigcap_{g \in x_n v f} \phi_{A'}(g).$$

As $(\forall x_n A')^f \in a$, $a \in ||(\forall x_n A')^f||$, so $||(\forall x_n A')^f||$ is the desired member of Prop.

For CNegUnivSub, suppose $a \in \phi_{s=t}(f)$ and $a^* \in \phi_{\forall x_n A'}(f)$, for $\rightarrow$-free A'. From the assumption, $(s = t)^f \in a$ and $(\forall x_n A')^f \in a^*$. Suppose that $a^* \notin ||(\forall x_n A)^f||$. Then, $(\forall x_n A)^f \notin a^*$. So, $(\sim\forall x_n A)^f \in a$. By (I3), it follows that $(\sim\forall x_n A')^f \in a$. But then, $(\forall x_n A')^f \notin a^*$, so $a^* \notin ||(\forall x_n A')^f||$, which implies $a^* \notin \phi_{\forall x_n A'}(f)$, contradicting an assumption. Therefore, $a^* \in ||(\forall x_n A)^f||$. By Lemma 16,

$$||(\forall x_n A)^f|| \subseteq \bigcap_{g \in x_n v f} \phi_A(g).$$

Thus, $||(\forall x_n A)^f||$ is the desired member of Prop. □

With that lemma, we then claim Completeness for QR$^{=}$.

Theorem 9 (Completeness) *For all formulas A, if* $\models_{\mathsf{QR}^=} A$, *then* $\vdash_{\mathsf{QR}^=} A$.

Proof The proof is that of [33]. Suppose $\models_{\mathsf{QR}^=} A$. It follows that in the canonical model, $A^f \in a$, for all $a \in O$ and all $f \in I^\omega$. Let $x_1, \ldots, x_n$ be exactly the free variables of A. Pick an assignment $g \in I^\omega$ such that $g(x_1) = c_1, \ldots, g(x_n) = c_n$, where all the c_i's are distinct and none otherwise occur in A. Then $A^g \in a$, so $\vdash_{\mathsf{QR}^=} A^g$. It follows that $\vdash_{\mathsf{QR}^=} \forall x_1 \ldots \forall x_n A$, so $\forall x_1 \ldots \forall x_n A \in a$, as $a \in O$. From (Q1), $A \in a$, whence $\vdash_{\mathsf{QR}^=} A$, as desired. □

The conditions on $\mathsf{QR}^=$-models, CUnivSub and CNegUnivSub, may strike some as unappealing, as there is an apparent intrusion of syntax into the model conditions. At present, we do not see a way around this, provided the target logic is $\mathsf{QR}^=$. If one is willing to strengthen the logic to $\mathsf{QR}^=_{sub}$, with the addition of (I4), then the situation is different. Models for $\mathsf{QR}^=_{sub}$ can drop CUnivSub and CNegUnivSub, instead using the condition CFullSub. Call the $\mathsf{QR}^=$-models satisfying CFullSub strong $\mathsf{QR}^=$-models. The logic $\mathsf{QR}^=_{sub}$ is sound and complete with respect to the class of all strong $\mathsf{QR}^=$-models.

Theorem 10 *For all formulas A, $\models_{\mathsf{QR}^=_{sub}} A$ iff $\vdash_{\mathsf{QR}^=_{sub}} A$.*

Proof We will prove the new cases for (I4) the soundness direction. Assume $f, a \Vdash_V A$ and $f, a \Vdash_V s = t$. Let $Vf(s) = i$ and $Vf(t) = j$, so $(i, j) \in \approx_a$, and so a is (i, j)-conflating. The first case is when A is of the form $\forall x B$. Suppose that $f, a \nVdash_V \forall x B'$. Then there is a formula C that is a skeleton for both $\forall x B$ and $\forall x B'$. As in the proof of Theorem 5, we will assume B and B' differ on one term occurrence for notational simplicity, so $B = C[x_{n_1}/t_1, \ldots, x_{n_m}/t_m, x_{n_{m+1}}/t]$ and $B' = C[x_{n_1}/t_1, \ldots, x_{n_m}/t_m, x_{n_{m+1}}/s]$. Let g be the assignment $f[j_1/n_1, \ldots, j_m/n_m]$, where $Vf(t_i) = j_i$ for $1 \le i \le m$. As $f, a \Vdash_V \forall x B$, by Lemma 12, $g[i/n_{m+1}, a \Vdash_V \forall x C$. Again by Lemma 12, $g[j/n_{m+1}], a \Vdash_V \forall x C$ iff $f, a \Vdash_V \forall x B'$. So, $a \notin \phi_{\forall x C}(g[i/n_{m+1}])$ while $a \in \phi_{\forall x C}(g[j/n_{m+1}])$. As a is (i, j)-conflating, we have a contradiction, since $g[i/n_{m+1}] \sim^i_j g[j/n_{m+1}]$. Therefore, $f, a \Vdash_V \forall x B'$. The case where A is of the form $\sim\forall x B$ is similar.

Nothing essentially new is needed the completeness proof, which piggybacks on the work that has already been done. The proof that the canonical frame satisfies the condition CFullSub is basically the same as in the earlier section, and similarly for the model conditions. □

To conclude this section, I will note that the axiom (EC), $\forall x(A \vee B) \to A \vee \forall x B$ where x is not free in A, can be added to either $\mathsf{QR}^=$ or $\mathsf{QR}^=_{sub}$. Models for those logics can be obtained by adding the following model condition, provided by [33],

$$X \backslash Y \subseteq \bigcap_{g \in x_n v f} |B|_{Vg} \Rightarrow X \backslash Y \subseteq |\forall x B|_{Vf},$$

for any formula B, any $X, Y \in$ Prop, and any $f \in I^\omega$. Soundness and completeness results can be obtained using the arguments similar to those of [33].

7 Discussion

I have provided models for logics of identity over the base logic R, with and without quantifiers. It is worth comparing the resulting models with those of Mares [29], whose models are based on the models of Fine [19]. This leads to a rough distinction between local and global features of the models. I will then close with a discussion of constant and varying domain models for quantified relevant logics of identity.

There are several differences between the present models and the Fine-Mares models for identity. First are the differences, of which there are many, in the underlying models for relevant logics. While I will not provide all the details of the Fine-Mares models, leaving the interested reader to consult the original texts, there is one difference I will discuss, namely the treatment of quantification. Fine's approach to quantification uses something like arbitrary objects and the Mares-Goldblatt approach uses propositions.[17] The truth condition for the universal quantifier in the Fine-Mares models is the following.

$$a \Vdash \forall x B \text{ iff } (\exists a\uparrow)(\exists i \in D_{a\uparrow} - D_a)(\tau(u) = i \,\&\, a\uparrow \Vdash B[^u_x])$$

A universally quantified formula is true at a point a iff the point, $a\uparrow$ is a point whose domain, $D_{a\uparrow}$ extends that of a, D_a, with a new object, i, that the interpretation τ assigns to the variable u. Whether a universal is true at a point a then requires checking a single, special instance of the quantified formula at the point $a\uparrow$.

To model identity, the Fine-Mares models use a family of binary relations on the domain that may vary from point to point. These have several conditions on them, analogs of (ID1)–(ID7).[18] The differences in the models have some bearing on the treatment of identity. Let us start with the treatment of (I3). The Fine-Mares models do not need an additional condition to validate (I3) beyond the basic conditions on the family of binary relations when quantifiers are in the language.

In Fine-Mares models, the truth of a universally quantified formula at a point depends on features that are plausibly *local* to that point, namely its extended domain points, and there is a condition in the Fine-Mares models that says that $\approx_a \subseteq \approx_{a\uparrow}$.[19] Because of this, there are no additional conditions needed in order to validate the substitution of identicals in the scope of a universal quantifier, or a negated universal.[20] In contrast, the truth of a universally quantified formula in the Mares-Goldblatt models depends on a *global* feature of the model, namely the set Prop and the truth

[17] Fine [19] sketches an interpretation of the models that is close to the generic semantics of [16–18]

[18] Indeed, the conditions adopted in this paper were based on those used for the Fine-Mares models.

[19] The Fine-Mares models have an array of conditions relating $\uparrow$, $^{\rightarrow}$, the other parts of the model, including the operation $\uparrow$ and the relation $\leq$, e.g. $a \leq b \Rightarrow a\uparrow \leq b\uparrow$, which are global constraints on interactions between these features of the models. There is more to say in for a full development of the local/global distinction for Fine-Mares models, which would be worthwhile, but it is beyond the scope of this paper and these remarks. I would like to thank an anonymous referee for pointing out the subtleties involved here.

[20] The interested reader is encouraged to look at [29, 14], in particular case 4 for axiom 13 in the proof of theorem 3.1. Since in the Fine-Mares models, the truth of a universal at a point b is equivalent to the truth of an instance in $b\uparrow$, substitution in the scope of a universal quantifier is secured via substitution on that instance. Substitution in the scope of a negated universal quantifier is similar, albeit involving interactions between $\uparrow$, $\downarrow$, $\leq$, and $-$, which is used for the truth condition for negation.

sets. Whether $\forall x Fxs$ and $\forall x Fxt$ hold at a point, at which $s = t$ holds, depends on the truth sets of all the instances of the quantified formulas, and those truth sets may contain points at which $s = t$ fails. While the truth sets for $\forall x Fxs$ and $\forall x Fxt$ overlap, by stipulation, they need not coincide. There needs to be additional conditions in order to ensure that substitution of identicals in the scope of a universal quantifier is sound.

Let us turn to (I4). The condition Mares places on the Fine-Mares models to validate (I4) is the following.

CldClosure For distinct i and j, if $(i, j) \in \approx_a$, then $a = \overset{i,j\rightarrow}{a}$.[21]

In this condition, $\overset{i,j\rightarrow}{a}$ is "the smallest [point] containing a in which $A(u)$ is true if and only if $A(v)$ is true, where 'u' refers to i and 'v' refers to j."[22] It is clear why CldClosure validates (I4): It says that if a point considers two objects identical, then no formula can distinguish them at that point. The condition CFullSub likewise ensures the validity of the substitution of identicals in arbitrary formulas, albeit by placing a condition on assignments and PropFun.

Mares's CldClosure is, plausibly, a local condition, since verifying it requires checking each point a to see whether it takes i and j to be the same and if so, checking that the $\overset{i,j\rightarrow}{a} = a$. There will, of course, be consequences to $\overset{i,j\rightarrow}{a} = a$ being true that will reach beyond the point a, namely it will constrain the evaluation of the antecedent and consequent of conditionals at other points, even if the identity between i and j does not hold at those other points. CFullSub, by contrast, is plausibly a global condition, as it places constraints on PropFun, rather than a local condition, as with the Fine-Mares models.

A global condition along the lines of CFullSub seems like the only way to validate (I4) in the Mares-Goldblatt models. An identity holding at one point needs to affect the evaluation of formulas being evaluated at other points, possibly several steps down a chain of R-related points, without requiring that the identity hold at the other points. While we cannot explain all the differences between the Fine-Mares models and the Mares-Goldblatt models in terms of a local-global distinction, the distinction can give us some traction on important differences in how identity interacts with the quantifiers.

To close, I will comment on one other feature of the models: The Mares-Goldblatt models are constant domain models in the sense that the domain, I, is the same from point to point.[23] In contrast, the Fine-Mares models are sometimes criticized for being varying domain models, in the sense that different points may have distinct domains.[24] To the extent that one sees the varying domains of the Fine-Mares models as a problem, the Mares-Goldblatt models will be more attractive. Yet, care should be taken in discussing constant domains in this context, as some issues arise.

[21] This condition is (VI.x′) in [29]. The name has been changed to match the conventions of this paper.

[22] Mares [29, 5].

[23] Mares [31] shows how to accommodate varying domains in Mares-Goldblatt frames.

[24] See Logan [28] for discussion of this varying domain objection.

The Mares-Goldblatt models are constant domain, despite the fact that the (EC) axiom, $\forall x(A \vee B) \rightarrow A \vee \forall x B$, where x is not free in A, which is valid in constant domain models for quantified intuitionistic logic, is not valid in Mares-Goldblatt models.[25] As noted, (EC) is an optional extra for Mares-Goldblatt models that can be added to QR to obtain the logic RQ. (EC) is valid in the varying domain Fine-Mares models. In models for relevant logics, (EC) does not neatly track the distinction between constant and varying domain models. There is a further subtlety.

Once identity is in the language, there is a sense in which varying domains arise. Let us call the domain, I, of a Mares-Goldblatt model the *global domain*. We can then define a *local domain*, I_a, for each point $a \in K$, as follows. Let $[i]_a = \{j \in I : (i, j) \in \approx_a\}$, and let $I_a = \{[i]_a : i \in I \,\&\, \exists j \in I((i, j) \in \approx_a)\}$. The local domain I_a will be the set of non-empty sets $[i]_a$. From the closure conditions on $\approx_a$, if $j \in [i]_a$ and $k \in [i]_a$, then $[j]_a = [k]_a = [i]_a$. Following Goldblatt [21, 162], we can define an existence predicate, E, as Et iff $t = t$. Existence claims will be true at a point (at an assignment) just when the terms denote objects in the local domain equivalence classes. When CFullSub is adopted, for $i, j \in [k]_a$, $A(i)$ will hold at a iff $A(j)$ does. There is more to explore regarding local domains, but the feature to focus on is that the local domains can vary from point to point.[26] This is true even when one is going along the heredity ordering.[27] If $a \leq b$, then there may be an i such that $[i]_b \in I_b$ but $[i]_a \notin I_a$. Additionally, one may have i and j such that $[i]_a \neq [j]_a$ while $[i]_b = [j]_b$.

The upshot is that local domains need not be constant. Proceeding along the heredity ordering, points can recognize new objects, increasing their local domain over their hereditary predecessors. They can also conflate objects that were previously distinct. This is so, even though the interpretation of names in Mares-Goldblatt models does not change from point to point. This flexibility may provide a distinctive way for the relevant logician to model and to respond to puzzles about contingent identity.[28]

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[25] See Moschovakis [34] or Priest [37, ch. 20] for quantifiers in Kripke models for intuitionistic logic.

[26] This definition of a local domain has some odd features; for example, one may have $a \Vdash Gs$, with $V(s) = i$, even though $[i]_a \notin I_a$.

[27] Varying of internal domains can happen in models for quantified intuitionistic logic with identity as well. See van Dalen [8], especially p. 35–37 and 61–64, for more on identity in Kripke models for intuitionistic logic. I would like to thank an anonymous referee for pointing this out to me.

[28] See Schwarz [42] for an overview of contingent identity.

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