FAILURES OF γ

TORE FJETLAND ØGAARD[1] AND SHAWN STANDEFER[2,†]

Abstract:
Since Meyer and Dunn showed that the rule γ is admissible in **E**, relevantists have produced new proofs of the admissibility of γ for an ever more expansive list of relevant logics. We show in this paper that this is not cause to think that this is the norm; rather γ fails to be admissible in a wide variety of relevant logics. As an upshot, we suggest that the proper view of γ-admissibility is as a coherence criterion, and thus as a selection criterion for logical theory choice.

Keywords: · coherence · γ-admissibility · relevant logic

1. Hard work and a clean life

One of the more radical elements of the philosophy of relevant logics has been and continues to be the rejection of the validity of disjunctive syllogism—the rule to the effect that B is obtainable from $\neg A$ together with $A \vee B$, or equivalently given minimal rules for negation, that B is obtainable from A together with $\neg A \vee B$. Since the material conditional $A \supset B$ is definable as $\neg A \vee B$, the latter form, then, is merely modus ponens for the material conditional.[1] The relevant rejection of this, then, amounts to a flat out rejection of the material conditional as a proper conditional. In Anderson and Belnap's playful style: " 'Material implication' is *not* a 'kind' of implication, or so we hold; it is no more a kind of implication than a blunderbuss is a kind of buss" [Anderson and Belnap, 1962, 21].

The rule of modus ponens for the material conditional is known amongst relevant logicians as the rule of γ, a name given to the rule in what was one, if not *the*, *primum movens* of the enterprise of relevant logics, namely Ackermann [1956].[2] Indeed, the favorite logic of Anderson and Belnap—the logic **E** of *entailment*—is obtained from

[1]Department of Philosophy, University of Bergen, Norway
[2]Department of Philosophy, National Taiwan University, Taiwan
[†]Corresponding author
E-mail addresses: `Tore.Ogaard@uib.no`, `standefer@ntu.edu.tw`.
Date: May 13, 2025.

[1]For more on the philosophy of relevant logics, see Anderson and Belnap [1975], Read [1988], Dunn and Restall [2002], Mares [2004], Logan [2024], and Standefer [2024], among others.

[2]Urquhart [2016] provides an overview of three different methods that have been used by relevant logicians for showing that γ is admissible.

Ackermann's logic Π' of *rigorous implication* by simply dropping the rule γ. Even though, then, γ fails to be a derivable rule of inference in **E**, the question whether γ is *admissible* in **E** was raised as the first of several worth while open question in Anderson's *Some open problems concerning entailment*: "it might still be the case, by a sort of lucky accident, that whenever A and $\bar{A} \vee B$ are both *provable* in E, then B is also."[3] Anderson noted, then, that since (a) and (b) below are logical theorems of **E** that so would (c) be if γ was indeed admissible. In order to illustrate the two difficulties Anderson saw in proving γ to be admissible, Anderson displayed the following three logical theorems of **E**:

(a) $\neg(A \to A) \vee \big(\neg(B \to C) \vee \neg((A \wedge C) \to D) \vee ((A \wedge B) \to D)\big)$
(b) $A \to A$
(c) $\neg(B \to C) \vee \neg((A \wedge C) \to D) \vee ((A \wedge B) \to D)$

(a) through (c) are indeed all logical theorems of **E**. However, Anderson seems to have thought that if γ indeed was admissible, then (c) would follow *from* (a) and (b) in some sense, but noted that (c) seems to require the distribution axiom, whereas neither of (a) and (b) do.[4] "Worse, there is no clear relation between the proofs of (a) and (b), on the one hand and (c) on the other." Anderson noted that Belnap or he himself could most likely find a proof of B if presented with a provable A and $\neg A \vee B$, but wondered if there is a more general principle involved than what Belnap had suggested, namely "hard work and a clean life." There have not been many answers to this question offered in the literature. The only one we are aware of is by Weiss [2020], who provides a positive answer for relevant logics formulated in tableau systems, showing that the admissibility of the cut rule for tableau can be used to combine proofs of $\neg A \vee B$ and of A to get a proof of B.[5] As far as we know, no satisfactory answer has been given to this latter part of Anderson's query for other presentations of relevant logics, including Hilbert axiom systems, although the relevant research literature has become rife with proofs showing that γ is indeed admissible in various relevant logics (Meyer and Dunn [1969], Meyer et al. [1974], Meyer [1976], Mares and Meyer [1992], Seki [2011a,b, 2012], Weiss [2020], Dunn [2022], Kripke [2022]). There are, then, two parts to Anderson's γ problem which can be stated for any given logic **L**:

- Is γ admissible in **L**?
- If γ is admissible in **L**, is there a linking property connecting the proof of A and the proof of $\neg A \vee B$ to the proof of B, for any given A and B?

[3] Anderson [1963, 10]. Anderson and Belnap typically used '$\bar{A}$' as the negation of the formula A. In this paper we'll use '$\neg A$.'

[4] See Humberstone [2010] for discussion of Anderson and Belnap on admissible rules, albeit focused on a different rule than γ.

[5] Connections between cut and γ admissibility are also mentioned by Dunn and Meyer [1989] and Urquhart [2016]. We note that admissibility of cut is not, on its own, sufficient for γ admissibility, as the sequent system for **LR**, **R** without distribution, has cut admissible but γ is not admissible, as shown by Thistlewaite et al. [1988]. We thank an anonymous referee for drawing our attention to the Weiss and Urquhart references.

Anderson and Belnap came to believe that the answer to the second question in relation to **E** was *no*, that the admissibility of γ was indeed by "lucky accident":

> a much luckier accident, in view of the complication of the proof. The Meyer-Dunn argument [...] guarantees the existence of a proof of B, all right, but there is *no* guarantee that the proof of B is related in any sort of plausible, heartwarming way to the proofs of A and $\bar{A} \vee B$. [Anderson and Belnap, 1975, 299]

In this paper we aim not to overturn this judgement, but rather to throw some light on the flip-side of Anderson's follow-up question, namely the following question:

> Is there a unifying principle which can reasonably be singled out as the culprit in cases where γ *fails* to be admissible?

We have already seen that Anderson pointed to distribution as a possible source of γ failure. Indeed, Anderson's intuition turned out to be on the mark as it was verified by Meyer and Dunn [1969, 472] that $(A \wedge (B \to B)) \vee \neg A$ fails to be a logical theorem of **LE**—**E** without the distribution axiom—even though $\neg(B \to B) \vee ((A \wedge (B \to B)) \vee \neg A)$ and $B \to B$ are. Which logical principle of **LE**, then, is the culprit of the failure of γ? Note, first of all, that **LE**'s γ failure generalizes to even **LBX**—**B** without the distribution axiom but augmented by excluded middle. The three principles of **LBX** which seem reasonable to utilize in deriving $\neg(B \to B) \vee ((A \wedge (B \to B)) \vee \neg A)$ are

- Excluded middle
- De Morgan: substitute $\neg C \vee \neg D$ for $\neg(C \wedge D)$ in any formula
- Associativity of $\vee$

Associativity of extensional disjunction seems beyond reproach, and so we are left with two possible culpable sources: excluded middle and De Morgan. We know of no result showing that γ admissibility can be restored by giving up the De Morgan law. With regards to excluded middle, however, it is worth pointing out that γ is admissible in the contraction- and excluded middle-free logics **B**, **DW**, **TW** and **RW**.[6] However, we do not know whether this extends to the distribution free fragments of these logics.

Another source of failure was pointed to in what we take to be the first and only previous attempt at answering the above question, namely Meyer et al. [1984]. The very first proof of the admissibility of γ for **E** was presented by Meyer and Dunn [1969]. As is therein explained, the proof holds not only for **E**, but also for some related logics such as the weaker logic **T** of *ticket entailment*, and the stronger logics **R** and **RM**. Meyer and Dunn [1969] note that the transitivity and contraction axioms of **E** can be weakened somewhat while retaining the proof, and speculate that this might also be the case for the negation principles, but content with simply noting that the proof made use of "the DeMorgan Laws, excluded middle, and contraposition as well as unrestricted double negation" [Meyer and Dunn, 1969, 473]. Meyer et al. [1984] significantly sharpen this claim, however:

[6] This follows rather easily from the fact that these logics are prime [cf. Slaney, 1987].

> So it is Old Hat that distribution failures can lead to γ-failures. But what about *contraction* and *reductio* failures (and, for that matter, *exluded middle* failures, [...])? It is New Hat here that, in certain circumstances, the absence of members of this second group of principles will also lead to γ failures, even if (indeed, especially if) other laws, like distribution, are operating with full classical force. [Meyer et al., 1984, 250]

The main contribution of the latter paper, then, was in showing that γ fails to be admissible in the logic **CRW**—the contraction- reductio- and excluded middle-free logic **RW** augmented by Boolean negation.[7] Their proof depends on three factors:[8]

(1) $A \vee \neg^{\flat} A$ is a logical theorem
(2) $A \rightarrow A$ is a logical theorem
(3) $\neg^{\flat}\neg(A \rightarrow A)$ is not a logical theorem

In this paper we expand upon the result of Meyer et al. [1984], but whereas they made use of what we take to be a relevantly *impermissible* connective, namely Boolean negation,[9] we show that γ fails in less dramatic settings than the distribution-less and Boolean but contraction- and excluded middle-free ones previously pointed to. Rather, γ fails to be admissible in logics such as the weak relevant logic **B** and stronger relevant logics, such as **E**, if these are but augmented by a certain disjunctive axiom which itself is a logical theorem of the relevant logic **R**. Thus failure of γ admissibility for relevant logics need not in any recognizable way be due to failure of distribution or contraction, reductio, or excluded middle. One natural thought is that γ fails to be admissible due to some form of weakness—after all, such a logic fails to have some B as one of its theorems, even though both A and $\neg A \vee B$ are. There is some truth to this, but not in the flatfooted sense expressed by Jago [2013, 535]: "What would be interesting is to discover just how weak a relevant logic needs to be before disjunctive syllogism becomes inadmissible."

Paraconsistent relevant logics generally allow even logical truths to be dialetheias.[10] The question of the admissibility of γ is not so trivially reduced to a question of strength. As we'll see, γ can fail to be admissible in weak and strong relevant logics alike. It has also been shown to be admissible in weak and strong logics alike. We rather view the

[7]Note that **CRW** is a conservative extension of **RW**, as proved by Giambrone and Meyer [1989]. Cf. Restall [1993, 510].

[8]'$\neg^{\flat}$' is here the Boolean negation.

[9]See Standefer [2022] for a discussion of what a relevant connective is.

[10]Perhaps this is what Brady [2017, 303] is gesturing at when he says that γ, $\neg A, A \vee B \Vdash B$, "depends on the consistency of A." Taken literally, this is not generally correct, as we will see that, except for the connexive extension of **E**, all the failures offered for γ involve consistent formulas A.

admissibility of γ as a non-trivial criterion of coherence:[11] a provable material conditional ought to flow from the relevant conditional in some sense that we, alas, have so far yet to make precise sense of. Material implication may not generally be a kind of implication. A logically true one, however, is. In this paper it is therefore highlighted how easily γ can be made to fail regardless of the overall strength of the logic. Indeed, we will provide some ways to obtain failures for a wide range of relevant logics. The examples of γ failure will all be such as to highlight the axiom responsible for the provable disjunction and thus also the incoherence of the axiomatic construction of the logic.

The plan for the paper is as follows. In section 2, we will present the main axioms and relevant logics we will consider. In section 3, we will present our main results and demonstrate failures of γ for a wide range of relevant logics. We begin in §3.1 with failures that involve a particular axiom, valid in a well-known extension of **R**. Then we will demonstrate failures of γ in some well-known four-valued logics (§3.2), and then we turn to failures of γ in sublogics of **R** (§3.3). Finally, in section 4, we will conclude with some general lessons regarding failures of γ.

2. Logics and their axioms and rules

The weakest logic that we'll consider in detail in this paper will be the weak relevant logic **B**. This logic, as well as some of the more known relevant (and some quasi relevant) logics are presented in table 1. We will use the notation $\mathbf{L}[A_1, \ldots, A_n]$ for the logic **L** extended with the axioms $(A_1), \ldots, (A_n)$. For the purposes of this paper, we are considering logics in the framework FMLA, that is, as sets of formulas.[12] We require logics to be closed under substitutions, defined in the usual way.

[11]This is akin to taking Halldén completeness to be a criterion of reasonableness for disjunction in a logic. Halldén completeness is the property that if $A \vee B$ is a theorem of **L** with no atoms shared between the disjuncts, then A is a theorem or B is. Halldén completeness has been called 'Halldén reasonableness', as noted by van Benthem and Humberstone [1983].

[12]See [Humberstone, 2011, 103ff.].

(A1)	$A \to A$	
(A2)	$A \to A \vee B$ and $B \to A \vee B$	
(A3)	$A \wedge B \to A$ and $A \wedge B \to B$	
(A4)	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$	
(A5)	$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$	
(A6)	$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$	
(A7)	$\neg\neg A \leftrightarrow A$	
(A8)	$(A \to B) \to (\neg B \to \neg A)$	contraposition
(A9)	$(A \to B) \to ((C \to A) \to (C \to B))$	prefixing
(A10)	$(A \to B) \to ((B \to C) \to (A \to C))$	suffixing
(A11)	$A \to ((A \to B) \to B)$	assertion
(A12)	$(A \to (A \to B)) \to (A \to B)$	contraction
(A13)	$(A \to \neg A) \to \neg A$	reductio
(A14)	$((A \to A) \wedge (B \to B) \to C) \to C$	**E** axiom
(A15)	$A \to (A \to A)$	mingle
(A16)	$A \vee (A \to B)$	**3** axiom
(R1)	$\{A, B\} \Vdash A \wedge B$	adjunction
(R2)	$\{A, A \to B\} \Vdash B$	modus ponens
(R3)	$\{A \to B\} \Vdash (B \to C) \to (A \to C)$	suffixing rule
(R4)	$\{A \to B\} \Vdash (C \to A) \to (C \to B)$	prefixing rule
(R5)	$\{A \to B\} \Vdash \neg B \to \neg A$	contraposition rule

In order to substantiate a claim of non-derivability or non-theoremhood we will typically display an algebraic model for the logic in question in which the rule γ fails to be truth-preserving or the formula in question fails to be true. We will then display the

B	A1–A7, R1–R5	**DW**	**B** +A8 −R5
TW	**DW** +A8 −R5	**T**	**TW** +A12, +A13
E	**T** +A14	**R**	**T** +A11
RW	**TW** +A11	$\mathbf{R}_3$	**R** +A16
RM	**R** +A15	$\mathbf{RM}_3$	**RM** +A16

Table 1. Relevant logics

$$\begin{tikzcd}
& & & & & \mathbf{RM} \arrow[r] & \mathbf{RM}_3 \\
& & \mathbf{T} \arrow[r] & \mathbf{E} \arrow[r] & \mathbf{R} \arrow[r] \arrow[ur] & \mathbf{R}_3 \arrow[ur] & \\
\mathbf{B} \arrow[r] & \mathbf{DW} \arrow[r] & \mathbf{TW} \arrow[r] \arrow[u] & \mathbf{RW} \arrow[ur] & & &
\end{tikzcd}$$

Figure 1. The sublogic relation of the logics in table 1.

Hasse diagram of the partial order which conjunction and disjunction are interpreted as, respectively, greatest lower bound and least upper bound over. The other connectives are evaluated according to the displayed matrices. The subset $\mathcal{T}$ is the set of designated elements. A formula is true in such a model just in case it is evaluated to one such designated element, and a rule holds in a model just in case it is truth preserving. A formula is valid in a matrix if it is true in all models on that matrix, and a counterexample is a model in which a formula is not true. Unless otherwise stated, such models have all been found using Slaney's `MaGIC`—an acronym for *Matrix Generator for Implication Connectives*—which is an open source computer program created by John K. Slaney [1995].

3. Relevant failures of γ

Most relevant logics are sublogics of classical logic. There are some logics, however, which are contra-classical in the drastic way that the set of logical theorems would include every formula if it be closed under γ. One such example is the logic set forth by Routley and Meyer [1976], which includes a contradiction amongst its logical axioms. Most failures of γ are, however, due to the presence or absence of less dramatic axioms and theorems.

To begin, there is a simple, general theorem for γ admissibility that will help constrain the logics we will consider.[13]

Theorem 3.1. *Let* $\mathbf{L}$ *be consistent and prime. Then,* γ *is admissible in* $\mathbf{L}$.

Proof. Suppose that A and $\neg A \vee B$ are both theorems of $\mathbf{L}$. Since $\mathbf{L}$ is prime, either $\neg A$ is a theorem or B is a theorem. The first option contradicts the consistency of $\mathbf{L}$, so B is a theorem. □

This theorem gives a sufficient condition for γ admissibility. The conditions of the theorem are not necessary, as γ is admissible in $\mathbf{R}$, which is not prime. All the logics that we are considering are sublogics of classical logic, so they are, therefore, consistent. The main questions, then, will deal with logics that are not prime. We hasten to add that we are focusing on logics, rather than theories with non-logical axioms. A theory over a consistent logic need not itself be consistent, so γ admissibility for a logic does not transfer immediately to its theories.[14]

3.1. γ failures using the 3 axiom.

In their quest to show that there are stronger logics than $\mathbf{R}$ which satisfy the variable-sharing property, Routley et al. [1982, ch. 3.6] presented various axioms which are such that the resultant logic both satisfies the variable-sharing property, and γ fails to be admissible for it. One of these was the axiom (M9):

[13] A proof of the admissibility of γ in minimal logic using the primeness of the logic was given by Johansson [1936]. We thank an anonymous referee for directing our attention to this proof.

[14] It is worth noting, as demonstrated by Mangraviti and Tedder [2023], that there can be consistent theories over an inconsistent logic as well.

$((A \wedge \neg A) \to B) \vee (A \wedge \neg A)$ which is readily seen as a theorem in any logic with the **3** axiom. This, then, was the inspiration for the following theorem:

Theorem 3.2. *γ fails to be admissible in* $\mathbf{L}[A16]$*, where* $\mathbf{L}$ *is any logic between* $\mathbf{B}$ *and* $\mathbf{RM}_3$.

Proof. $\neg(A \to A) \vee (\neg(A \to A) \to B)$ is an instance of (A16) and so a theorem of $\mathbf{L}[A16]$. Since $A \to A$ also is, but $\neg(A \to A) \to B$ fails to be a logical theorem of $\mathbf{RM}_3$, the claim follows. □

Corollary 3.1. *γ fails to be admissible in* $\mathbf{R}_3$ *and in* $\mathbf{RM}_3$.[15]

$\mathbf{RM}_3$ fails to satisfy the variable-sharing property, although it does satisfy a weak version of this.[16] Note, however, that $\mathbf{R}_3$ *does* satisfy the variable-sharing property.[17] This follows from the fact that axiom (A16) is valid in the so-called *crystal lattice* displayed in fig. 2—one of the well-known algebraic models used to verify this property. This algebra was, to our knowledge, first presented by Routley et al. [1982, 250] and used therein to prove that the variable-sharing property holds true for various logics. Given the prominence of the crystal lattice, it is worth noting that γ fails to be admissible in the logic of the crystal lattice, which we will call **CL**. **CL** properly contains $\mathbf{R}_3$, but it still has a failure of γ, as we will see.

Using the crystal lattice, we can obtain a strong result on variable-sharing.

Theorem 3.3. *Any sublogic of* $\mathbf{L}$ *of* $\mathbf{R}_3$ *has the variable-sharing property.*

Proof. The proof is standard. See Routley et al. [1982, 250] for details. □

The failure of γ in theorem 3.2 actually tells us something more about any extension of $\mathbf{B}$ containing (A16).

Theorem 3.4. *Let* $\mathbf{L}$ *extend* $\mathbf{B}$*. If* $\mathbf{L}[A16]$ *is closed under* γ*, then* $\mathbf{L}[A16]$ *lacks the variable-sharing property.*

Proof. $\mathbf{L}[A16]$ has $\neg(p \to p) \vee (\neg(p \to p) \to q)$ as a theorem as well as $p \to p$. Closure under γ yields $\neg(p \to p) \to q$ as a theorem, but this is a violation of the variable-sharing property. □

Since having the variable-sharing property is typically taken as a necessary condition on being a relevant logic, this tells us that the combination of (A16) and γ cannot be enjoyed by any relevant logic, unlike the axioms considered in subsection 3.3, which can be combined with γ in a relevant logic. This theorem gives us a corollary on **CL**.

[15] This was first proven by Dunn [1970, 10] using the fact that $(p \wedge \neg p) \vee ((p \vee \neg p) \to (q \vee \neg q))$ and excluded middle are logical theorem of $\mathbf{RM}_3$.

[16] See Øgaard [2023] for more on the notion of weak variable-sharing for $\mathbf{RM}_3$.

[17] Standefer [2025] argues that satisfying the variable-sharing property, along with being closed under (R1) and (R2), is sufficient for being a relevant logic. We will work with this definition below, although the logics we are investigating are focusing on a relatively narrow portion of such logics.

Corollary 3.2. *γ fails to be admissible in* **CL**.

Proof. **CL** has the variable-sharing property and contains (A16). The result then follows from theorem 3.4. □

As **CL** is a proper superlogic of **R**, this gives another example of a strong relevant logic that lacks γ admissibility.

The logics we have considered so far are stronger than or incomparable with **R**. The most often discussed relevant logics fall between **B** and **R**. Before turning to logics in that range, we will look at three prominent extensions of the four-valued logic **FDE**.

3.2. γ **failures for four-valued logics.** The failures of γ so far have been due to (A16), the 3-axiom. It has some plausibility in the context of formulating a three-valued paraconsistent logic, although perhaps unappealing from the point of view of the relevant logician. This type of motivation is precisely what was given by Meyer et al. [1984] for the four-valued paraconsistent and paracomplete logic **BN4** which was shown by Brady [1982] to be the logic of the four-valued model displayed in figure 3 where T and F are to be interpreted as, respectively, the "just true" and "just false" truth values, and B and N as "both true and false" and "neither true or false" truth values.[18] **BN4** is one way to extend the four-valued logic **FDE** with a new implication connective. We will see three other ways below.

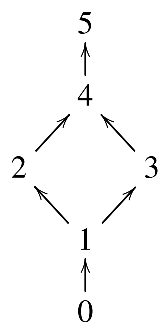
It is claimed by Meyer et al. [1984, 254] that it

> is evident that γ fails in **BN4**. For $B \vee N = T$ [...]. So, intuitively, both of $B \vee N$, $-B(= B)$ are (at least) true elements of the matrix. But N, since it is interpretable as neither true nor false, remains *undesignated*.

What is evident, however, is only that this is somewhat shy of a satisfactory proof of γ failure. The following proof rectifies this.

[18] A logic is regarded as paraconsistent if it is the case it allows for non-trivial inconsistent theories, and paracomplete if excluded middle fails to be a logical theorem.

$\mathcal{T} = \{1, 2, 3, 4, 5\}$



$\to$	0	1	2	3	4	5	$\neg$
0	5	5	5	5	5	5	5
1	0	1	2	3	4	5	4
2	0	0	2	0	2	5	2
3	0	0	0	3	3	5	3
4	0	0	0	0	1	5	1
5	0	0	0	0	0	5	0

FIGURE 2. The crystal lattice

Theorem 3.5. *γ fails to be an admissible rule in* **BN4**.

Proof. One of Brady's axioms for **BN4** is

$$A \lor (\neg A \to (A \to B)).$$

If we let A be $\neg(p \to p)$, and B be p, we obtain as one of the logical theorem of **BN4**

$$\neg(p \to p) \lor (\neg\neg(p \to p) \to (\neg(p \to p) \to p)).$$

Since $p \to p$ is a logical theorem of **BN4**, but the right-hand disjunct fails to be—it evaluates to N if p is evaluated to N—it follows that γ isn't admissible in **BN4**. □

Thus we have shown that γ fails in **BN4**, which complements the result of Meyer et al., who showed that γ fails in the extension of **BN4** with Boolean negation.[19]

Robles [2021] provides an alternative table for the arrow to obtain a sublogic of **BN4** that enjoys the variable-sharing property, which **BN4** does not. She calls it **BN4**^{vsp}. To obtain the arrow table for this logic, one changes the arrow table for **BN4** by setting $F \to B = N$ and $B \to T = N$. We will not dwell on the virtues of **BN4**^{vsp}, but rather note that it also exhibits failures of γ.

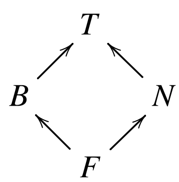
Theorem 3.6. *γ is not admissible in* **BN4**^{vsp}.

Proof. Robles [2021, 367] lists $(A \lor \neg B) \lor (A \to B)$ as an axiom of **BN4**^{vsp}. With some reassociating, this has as an instance, $\neg(p \to p) \lor (\neg(q \to q) \lor (\neg(q \to q) \to (p \to p)))$, setting B to $p \to p$ and A to $\neg(q \to q)$. Since $A \to A$ is valid, two applications of γ would yield $\neg(q \to q) \to (p \to p)$ as a theorem. This, however, is a violation of the variable-sharing property, which **BN4**^{vsp} has. Therefore, γ is not admissible. □

Robles [2023] presents a relative of **BN4**^{vsp}, **BN4**^{ap}, that enjoys the variable-sharing property as well as the Ackermann property, which says that $A \to (B \to C)$ is a theorem only if A contains an occurrence of the arrow. The logic **BN4**^{ap} is also obtained by a modification of the arrow table for **BN4**. The modification is more involved than the previous one, so we will omit the table here, but we will show that γ fails for this logic.

[19] NB: Meyer et al. use '$\neg$' for Boolean negation and '$-$' for de Morgan negation, whereas we use '$\neg$' for de Morgan negation and '$\neg^\flat$' for Boolean negation. It is also worth noting that the failure of γ that Meyer et al. demonstrate is γ formulated with De Morgan negation as its displayed negation. The version of γ with the displayed negation as Boolean negation holds.

$\mathcal{T} = \{B, T\}$



$\to$	F	B	N	T	$\neg$
F	T	T	T	T	T
B	F	B	N	T	B
N	N	N	B	T	N
T	F	F	N	T	F

FIGURE 3. The **BN4** model

Theorem 3.7. *γ fails for* $\mathbf{BN4}^{ap}$.

Proof. Robles [2023, 476] lists the rule $A \wedge B \Vdash (\neg A \vee \neg B) \vee (A \to B)$ in the axiomatization of $\mathbf{BN4}^{ap}$. Let A be $p \to p$ and B be $q \to q$. Suppose that $\mathbf{BN4}^{ap}$ is closed under γ. As both $p \to p$ and $q \to q$ are theorems, it follows that $\neg(p \to p) \vee \neg(q \to q) \vee ((p \to p) \to (q \to q))$ is as well. Two applications of γ would then result in $(p \to p) \to (q \to q)$ being a theorem. This would violate the variable-sharing property, which $\mathbf{BN4}^{ap}$ has, therefore $\mathbf{BN4}^{ap}$ is not closed under γ. □

Robles and Méndez [2016] provide an alternative table for the arrow on the four-valued matrix above to obtain a logic they call **E4**. To obtain the table for the **E4** arrow, we change all the 'N's in the **BN4** table to 'F's. We can obtain a similar failure of γ for **E4**.

Theorem 3.8. *γ is not admissible in* **E4**.

Proof. As one of the theorems of **E4**, Robles et al. [2016, 97] list $(A \vee \neg B) \vee (A \to B)$. With some reassociating, an instance of this is $\neg(p \to p) \vee (q \vee (q \to (p \to p)))$, setting A to q and B to $p \to p$. The righthand disjunct, however, is not valid, as $v(p) = B$ and $v(q) = N$ provides a counterexample:

$$N \vee (N \to (B \to B)) = N \vee (N \to B) = N \vee F = N.$$

Therefore, γ is not admissible in **E4**.

□

All of **BN4**, $\mathbf{BN4}^{vsp}$, $\mathbf{BN4}^{ap}$, and **E4** are presented as natural extensions of the four-valued matrix for **FDE** with an implication connective. None, however, has γ admissible.

That is enough on the logics outside of the usual range of relevant logics, namely between **B** and **R**. We will now turn to failures of γ for logics in this range.

3.3. **γ failures for sublogics of R.** Except for $\mathbf{RM}_3$ and $\mathbf{R}_3$, γ is admissible in all the logics displayed in table 1.[20] In this subsection it will be shown that each of these sublogics of **R** can easily be augmented so as to yield a logic which retains the property of being a sublogic of **R**—and therefore also the property of variable-sharing—yet fails to have γ as an admissible rule. The idea here is simple: $\neg(A \to A)$ is satisfiable in any sublogic of **R** as it can be made true in the crystal lattice. We then find some B which is a logical theorem of **R** but which fails to be a logical theorem of $\mathbf{L}^{\dagger}$—**L** augmented by $\neg(A \to A) \vee B$—where **L** is a sublogic of **R**. If such a B can be found, we can conclude that γ fails to be admissible in $\mathbf{L}^{\dagger}$. We cannot turn this plan into a general theorem, because, as far as we can tell, there is not a general method of showing that $\mathbf{L}^{\dagger}$ fails to have B as a theorem, even though **L** without the additional axiom will fail to have B as a theorem. For particular choices of B, however, we can get the desired result.

[20] The proof of Meyer and Dunn [1969] covers **T**, **E**, **R**, and **RM**. That **B**, **DW**, **TW**, and **RW** admit γ was proven by Slaney [1987].

To avoid having to prove the same thing for each of the **R**-sublogics of table 1, we start by noting that the permuted version of the contraction axiom,

$$\text{(PC)} \quad A \to ((A \to (A \to B)) \to B)$$

is a theorem of **R**, yet fails to be a theorem of both **E** and of **RW** even when augmented with the following axiom "dialethic-disjunctive" version of the PC-axiom:

$$\text{(dPC)} \quad \neg(A \to A) \vee (A \to ((A \to (A \to B)) \to B))$$

Since (PC) is a logical theorem of **R**, (dPC) is so too. Thus adding (dPC) to any sublogic of **R** will still beget a sublogic of **R**. That PC fails in **E**[dPC] is verified by the model displayed in figure 4. That it fails in sublogics of **RW**[dPC] is verified by the model displayed in figure 5.

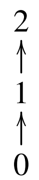
Theorem 3.9. *γ fails to be admissible in* **B**[dPC], **DW**[dPC], **TW**[dPC], **RW**[dPC], **T**[dPC], *and* **E**[dPC].

For the next example, we will need to define some notation. Let $\Box A$ be defined as $(A \to A) \to A$, which is how Anderson and Belnap define the necessity operator of **E**. Next, let (E1) be the axiom $\Box(A \to A)$ and let (E2) be the axiom $\neg\Box A \vee (\neg\Box(\neg A \vee B) \vee \Box B)$, both of which are theorems of **R**. With this in hand, we can state our next theorem.

Theorem 3.10. *Let* **L** *be any sublogic of* **E**. *Then* **L**[$E1, E2$] *fails to have γ admissible.*

$\mathcal{T} = \{1, 2\}$

$2 \to ((2 \to (2 \to 0)) \to 0) = 0$

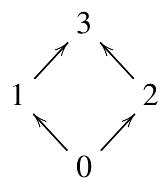


$\to$	0	1	2	$\neg$
0	1	1	1	2
1	0	1	1	1
2	0	0	1	0

FIGURE 4. An **E**[dPC] model

$\mathcal{T} = \{1, 3\}$

$2 \to ((2 \to (2 \to 1)) \to 1) = 2$



$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	0	1	2	3	1
2	2	2	1	3	2
3	0	0	0	3	0

FIGURE 5. A **RW**[dPC] model

Proof. To obtain a failure of γ, take the instance of (E2) with A as $p \to p$ and B as q and the instance of (E1) with A as p. Then, the left disjunct of (E2) is the negation of that instance of (E1). The right disjunct of the instance of (E2), namely $\neg\Box(\neg(p \to p) \vee q) \vee \Box q$ has a 12 element countermodel, as found by `MaGIC`. $\Box$

E has (E1) as a theorem, but many of its sublogics, notably **T**, do not. Therefore, we include it for additional generality. **E** does not have (E2) as a theorem. The intuition behind (E2) is that it is a form of distribution of necessity over the defined material conditional, which is valid in normal modal logics over classical logic.[21] That formula is the one that Mares [2000] uses to show that **E** is not conservatively extended by Boolean negation. The extension has an instance of (E2) as a theorem that is not a theorem of **E** without Boolean negation.

Another way to find a logic for which γ fails to be admissible is to rather focus upon the axioms of the logics which extend it. Let's look at **B** as an example. **DW** is obtained from this logic by replacing the contraposition rule by its axiomatic version (A8). We have already noted that γ is admissible in both these logics. As above it can be verified that adding

$$\text{(dA8)} \quad \neg(A \to A) \vee ((B \to C) \to (\neg C \to \neg B)))$$

to **B** yields a logic in which (A8) fails to be a logical theorem and which, therefore, fails to have γ as an admissible rule. In fact, we can strengthen this claim, which we will codify as a theorem. We will consider a logic that is, roughly, **E** without (A8).

Theorem 3.11. *Let* **L** *be any sublogic of* **B**$[A9, A10, A12, A13, A14]$. *Then γ fails to be admissible in* **L**$[dA8]$.

Proof. The logic **B**$[A9, A10, A12, A13, A14, dA8]$ is valid in the matrix in figure 6. Setting $v(q) = v(r) = 0$ yields

$$(0 \to 0) \to (\neg 0 \to \neg 0) = 2 \to (3 \to 3) = 2 \to 1 = 0$$

as desired. $\Box$

[21] This is the (K) axiom of normal modal logics. (E2) can be rewritten as $\Box A \supset (\Box(A \supset B) \supset \Box B)$.

$\mathcal{T} = \{1, 2, 3\}$

$\to$	0	1	2	3	$\neg$
0	2	2	2	2	3
1	0	1	2	2	2
2	0	0	2	2	1
3	0	0	0	1	0

FIGURE 6. A **B**$[A9, A10, A12, A13, A14, dA8]$ model

As a second example, adding $\neg(A \to A) \vee ((A \to \neg A) \to \neg A)$ to **TW** results in a proper sublogic of **TW**[$A13$] for which, then, γ fails to be admissible.[22] Lastly, Restall [1993] emphasized the fact that $\neg A \vee ((A \to B) \to B)$ fails to be a logical theorem of **E**, although it is provable in **R**. Note, then, that adding this axiom to **T** yields a logic for which γ fails to be admissible seeing as $((A \to A) \to B) \to B$ fails to be a theorem of the resultant logic, although it is a theorem of **E**.[23]

Thus, we have supplied some methods for finding failures of γ in sublogics of **R** and demonstrated such failures for a wide range of logics. It is worth noting that the examples can be multiplied. In the disjunctive axioms we have considered, the left disjunct was $\neg(A \to A)$, but there is nothing special about our choice of $A \to A$ here. Since relevant logics have non-equivalent theorems, one could choose a different, non-equivalent theorem of **B** to obtain further examples of failures.[24] As an example, in (dPC), one could use $\neg C$ as the left disjunct, where C is any theorem of **B** not equivalent to $A \to A$, and the formula $\neg C \vee (A \to ((A \to (A \to B)) \to B))$ would equally demonstrate the failures of γ akin to those of theorem 3.9.

4. Conclusions

We have seen that γ can fail in strong logics, including logics stronger than **R**, and we have seen that it can fail in logics weaker than **R**. We have seen that γ can fail in logics incomparable with **R**. γ can fail in logics with the variable-sharing property and in logics without the variable-sharing property. Indeed, based on the proposals for generating failures that we presented above, γ failing seems to be the norm, rather than the exception.

An analogy with classical logic and its normal modal extensions will be helpful to clarify our point. It is known that many of the standard normal modal logics are sound and complete with respect to some class of Kripke frames. This might give one the impression that completeness with respect to a class of frames is the norm. This is not the case, however, as modal incompleteness, failing to be complete with respect to any class of frames, is in fact the norm.[25] Could the situation with γ in relevant logics be similar to that of completeness with respect to a class of frames in normal modal logics? Although the results above do not entail this, they do suggest that the answer is yes, since the results of §3.3 give ways of systematically generating logics for which γ fails that are contained in **R**. This highlights the question of what is distinctive about

---

[22] That the sublogic relation here is a proper one can, as in the above case of **B**, easily be verified using `MaGIC`. Also: The question of whether γ is admissible in **TW**[$A13$] was raised by Robles and Méndez [2010] but remains, to our knowledge, unresolved.

[23] This is also a fact easily verified using `MaGIC`, a task that we leave to the reader.

[24] In the terminology of Standefer [2019], a logic is polythetic iff it has non-equivalent theorems. This is in contrast to monothetic logics, for which see Humberstone [2011, 221]. In monothetic logics, all thoerems are equivalent.

[25] See Blackburn et al. [2002, 260ff.] for discussion of this point.

the logics for which γ is admissible, a question for which we do not have an answer at present.

It is sometimes suggested that γ fails because of inconsistency. For non-logical theories, this seems plausible. The logics we discussed above, however, are all consistent, as they are sublogics of classical logic. The closest one gets to inconsistency is that there are matrices that give a designated value to some formula and its negation. Let us call such matrices *inconsistent*. Having an inconsistent matrix for which the logic is sound is not sufficient for a failure of γ, since γ is admissible for **R** and there are many inconsistent matrices for which **R** is sound. One might think that having a *characteristic* matrix that is inconsistent is sufficient for a failure of γ. This too is incorrect, as one can see by considering the three-valued logic **LP**.[26] One can get **LP** from the **BN4** matrix by dropping the implication from the language and considering only the models that assign values from the set $\{T, F, B\}$. γ is admissible in **LP**, since the theorems of **LP** are exactly those of classical logic.[27] As a second example, note that it follows from the result reported in Meyer [1971] that the inconsistent $\mathbb{Z}$-valued Sugihara matrix is characteristic of **RM**.[28] We tentatively conclude that inconsistency is not the proper diagnosis for failures of γ in the logics discussed above.

At the end of their article, Meyer et al. [1984, 255] say that proponents and critics of relevant logics "have been looking at the wrong systems. It is better that such issues should be joined where γ fails." Based on their examples, they mean logics with Boolean negation and logics without distribution, contraction, or reductio. We don't disagree that those are good examples to consider. They are not, however, the only examples, as we have shown that many logics with the highlighted principles have failures of γ. We would add to the sentiment of Meyer et al. that the failures of γ are widespread throughout the family of relevant logics.

Acknowledgments

We would like to dedicate this paper to the memory of Nuel Belnap. His work on relevant logics was ground-breaking and continues to be influential on the field, and on our research. We hope that he would have enjoyed this paper.

We would like to thank Ed Mares, Rohan French, Lloyd Humberstone, and two anonymous referees for feedback that helped to improve this paper. Shawn Standefer's research was supported by the National Science and Technology Council of Taiwan grant 111-2410-H-002-006-MY3.

[26]See Priest [1979] for more on **LP**.

[27]This fact is noted by Priest [1979].

[28]Expanding on Meyer's work on **RM**, Dunn [1970] showed that the inconsistent $\mathbb{Q}$-valued Sugihara matrix is even *strongly* characteristic of **RM**.

References

Wilhelm Ackermann. Begründung einer strengen Implikation. *Journal of Symbolic Logic*, 21(2):113–128, 1956. DOI 10.2307/2268750.

Alan R. Anderson and Nuel D. Belnap. *Entailment: The Logic of Relevance and Necessity*, volume 1. Princeton University Press, Princeton, 1975.

Alan Ross Anderson. Some open problems concerning the system E of entailment. *Acta philosophica Fennica*, 16:7–18, 1963.

Alan Ross Anderson and Nuel D. Belnap. The pure calculus of entailment. *Journal of Symbolic Logic*, 27(1):19–52, 1962. DOI 10.2307/2963676.

Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2002.

Ross T. Brady. Completeness proofs for the systems RM3 and BN4. *Logique et Analyse*, 25(97):9–32, 1982. URL `https://www.jstor.org/stable/44084001`.

Ross T. Brady. Metavaluations. *Bulletin of Symbolic Logic*, 23(3):296–323, 2017. DOI 10.1017/bsl.2017.29.

J. Michael Dunn. Algebraic completeness results for R-mingle and its extensions. *Journal of Symbolic Logic*, 35(1):1–13, 1970. DOI 10.1017/S0022481200092161.

J. Michael Dunn. Kripke's argument for γ. In Katalin Bimbó, editor, *Relevance Logics and other Tools for Reasoning: Essays in Honor of J. Michael Dunn*, pages 178–181. College Publications, 2022.

J. Michael Dunn and Robert K. Meyer. Gentzen's cut and Ackermann's gamma. In Jean Norman and Richard Sylvan, editors, *Directions in Relevant Logic*, volume 229–240. Kluwer, Dordrecht, 1989.

J. Michael Dunn and Greg Restall. Relevance logic. In Dov M. Gabbay and Franz Guenthner, editors, *Handbook of Philosophical Logic*, volume 6, pages 1–136. Kluwer, 2nd edition, 2002.

Steve Giambrone and Robert K. Meyer. Completeness and conservative extension results for some boolean relevant logics. *Studia Logica*, 48(1):1–14, 1989. DOI 10.1007/bf00370629.

Lloyd Humberstone. Smiley's distinction between rules of inference and rules of proof. In T. J. Smiley, Jonathan Lear, and Alex Oliver, editors, *The Force of Argument: Essays in Honor of Timothy Smiley*, pages 107–126. Routledge, 2010.

Lloyd Humberstone. *The Connectives*. MIT Press, 2011.

Mark Jago. Recent work in relevant logic. *Analysis*, 73(3):526–541, 2013. DOI 10.1093/analys/ant043.

Ingebrigt Johansson. Der minimalkalkul, ein reduzierter intutionistischer formalismus. *Compositio Mathematica*, 4(1):119–136, 1936.

Saul Kripke. A proof of gamma. In Katalin Bimbó, editor, *Relevance Logics and other Tools for Reasoning: Essays in Honor of J. Michael Dunn*, pages 261–265. College Publications, 2022.

Shay Allen Logan. *Relevance Logic*. Elements in Philosophy and Logic. Cambridge University Press, 2024. DOI 10.1017/9781009227773.

Franci Mangraviti and Andrew Tedder. Consistent theories in inconsistent logics. *Journal of Philosophical Logic*, 52(04):1133–1148, 2023. DOI 10.1007/s10992-023-09700-z.

Edwin D. Mares. **CE** is not a conservative extension of **E**. *Journal of Philosophical Logic*, 29(3):263–275, 2000. DOI 10.1023/A:1004731401855.

Edwin D. Mares. *Relevant Logic: A Philosophical Interpretation*. Cambridge University Press, 2004.

Edwin D. Mares and Robert K. Meyer. The admissibility of γ in $R4$. *Notre Dame Journal of Formal Logic*, 33(2):197–206, 1992. DOI 10.1305/ndjfl/1093636096.

Robert K. Meyer. R-mingle and relevant disjunction. *Journal of Symbolic Logic*, 36(2): 366, 1971. DOI 10.2307/2270323.

Robert K. Meyer. Ackermann, Takeuti, and Schnitt: For higher-order relevant logic. *Bulletin of the Section of Logic*, 5(4):138–142, 1976.

Robert K. Meyer and J. Michael Dunn. E, R and γ. *Journal of Symbolic Logic*, 34: 460–474, 1969. DOI 10.2307/2270909.

Robert K. Meyer, J. Michael Dunn, and Hugues Leblanc. Completeness of relevant quantification theories. *Notre Dame Journal of Formal Logic*, 15(1):97–121, 1974. DOI 10.1305/ndjfl/1093891202.

Robert K. Meyer, Steve Giambrone, and Ross T. Brady. Where gamma fails. *Studia Logica*, 43(3):247–256, 1984. DOI 10.1007/BF02429841.

Tore Fjetland Øgaard. The weak variable sharing property. *Bulletin of the Section of Logic*, 52(1):85–99, 2023. DOI 10.18778/0138-0680.2023.05.

Graham Priest. The logic of paradox. *Journal of Philosophical Logic*, 8(1):219–241, 1979. DOI 10.1007/BF00258428.

Stephen Read. *Relevant Logic: A Philosophical Examination of Inference*. Blackwell, 1988.

Greg Restall. Simplified semantics for relevant logics (and some of their rivals). *Journal of Philosophical Logic*, 22(5):481–511, 1993. DOI 10.1007/BF01349561.

Gemma Robles. A variant with the variable-sharing property of brady's 4-valued implicative expansion bn4 of anderson and belnap's logic fde. In Pietro Baroni, Christoph Benzmüller, and Y N. Wáng, editors, *Logic and Argumentation*, pages 362–376, Cham, 2021. Springer International Publishing. ISBN 978-3-030-89391-0.

Gemma Robles. Two 4-valued implicative expansions of first-degree entailment logic: The relevant logic BN4VSP and the (relevant) entailment logic BN4AP. *Journal of Logic and Computation*, 33(2):462–484, 01 2023. ISSN 0955-792X. DOI 10.1093/logcom/exac101.

Gemma Robles and José M. Méndez. A companion to Brady's 4-valued relevant logic BN4: The 4-valued logic of entailment E4. *Logic Journal of the IGPL*, 24(5):838–858, 2016. DOI 10.1093/jigpal/jzw011.

Gemma Robles and José M. Méndez. A Routley-Meyer semantics for relevant logics including TWR plus the disjunctive syllogism. *Logic Journal of the IGPL*, 19(1): 18–32, 2010. DOI 10.1093/jigpal/jzp083.

Gemma Robles, Sandra M. López, José M. Blanco, Marcos M. Recio, and Jesús R. Paradela. A 2-set-up Routley-Meyer semantics for the 4-valued relevant logic E4. *Bulletin of the Section of Logic*, 45(2), 2016. DOI 10.18778/0138-0680.45.2.03.

Richard Routley and Robert K. Meyer. Dialectical logic, classical logic, and the consistency of the world. *Studies in Soviet Thought*, 16(1):1–25, 1976. DOI 10.1007/BF00832085.

Richard Routley, Robert K. Meyer, Val Plumwood, and Ross T. Brady. *Relevant Logics and Their Rivals*, volume 1. Ridgeview Publishing Company, Atascadero, California, 1982.

Takahiro Seki. The γ-admissibility of relevant modal logics I — the method of normal models. *Studia Logica*, 97(2):199–231, 2011a. DOI 10.1007/s11225-011-9306-6.

Takahiro Seki. The γ–admissibility of relevant modal logics II — the method using metavaluations. *Studia Logica*, 97(3):351–383, 2011b. DOI 10.1007/s11225-011-9315-5.

Takahiro Seki. An algebraic proof of the admissibility of γ in relevant modal logics. *Studia Logica*, 100(6):1149–1174, 2012. DOI 10.1007/s11225-012-9459-y.

John Slaney. MaGIC, Matrix Generator for Implication Connectives: Release 2.1 notes and guide. Technical Report TR-ARP-11/95, Automated Reasoning Project, Australian National University, 1995. URL `http://users.cecs.anu.edu.au/~jks/magic.html`.

John K. Slaney. Reduced models for relevant logics without WI. *Notre Dame Journal of Formal Logic*, 28(3):395–407, 1987.

Shawn Standefer. Tracking reasons with extensions of relevant logics. *Logic Journal of the IGPL*, 27(4):543–569, 2019. DOI 10.1093/jigpal/jzz018.

Shawn Standefer. What is a relevant connective? *Journal of Philosophical Logic*, 51 (4):919–350, 2022. DOI 10.1007/s10992-022-09655-7.

Shawn Standefer. Routes to relevance: Philosophies of relevant logics. *Philosophy Compass*, 19(2):e12965, 2024. DOI 10.1111/phc3.12965.

Shawn Standefer. Variable-sharing as relevance. In Igor Sedlár, Shawn Standefer, and Andrew Tedder, editors, *New Directions in Relevant Logics*, pages 97–117. Springer, 2025.

Paul B. Thistlewaite, M. A. Mcrobbie, and Robert K. Meyer. *Automated Theorem-Proving in Non-Classical Logics*. Pitman Publishing, 1988.

Alasdair Urquhart. The story of γ. In Katalin Bimbó, editor, *J. Michael Dunn on Information Based Logics*, pages 93–105. Springer, 2016.

J. F. A. K. van Benthem and I. L. Humberstone. Halldén-completeness by gluing of Kripke frames. *Notre Dame Journal of Formal Logic*, 24(4):426–430, 1983. DOI 10.1305/ndjfl/1093870446.

Yale Weiss. Cut and gamma I: Propositional and constant domain R. *Review of Symbolic Logic*, 13(4):887–909, 2020. DOI 10.1017/s1755020319000388.