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FDE-based Modal Logics

Abstract. In this paper, we explore modal logics over **FDE** from the point of view of both the Australian plan models with the Routley star operator and the American plan models with four semantic values. We show that, in general, the Australian plan modal logics are weaker, although the addition of a frame condition allows us to prove an equivalence between the two approaches. We then consider different **S5**-type logics, using accessibility relations that are equivalence relations or universal relations, on the two different plans. We show that there are three different logics that result. Next, we provide sequent systems for all of the logics and show that two of the systems enjoy cut elimination. Then, we show soundness and completeness with respect to appropriate classes of models for all the logics. Finally, we discuss some features of the logics and close by highlighting two open problems.

Keywords: FDE, Modal logic, Star semantics, Australian plan, American plan, Universal modality.

1. Introduction

There are a number of interesting developments in modal logics having **FDE** as a core. [1] However, strictly speaking, these results are not for modal expansion of **FDE**. Rather, the base systems are expansions of **FDE** obtained by adding various kinds of conditional: classical, intuitionistic, or relevant. In this paper, we develop basic systems of modal logic with **FDE** as the basis, with some emphasis on the case of **S5**. There are two main precursors to our work.

First, there is the seminal investigation into **S5**-type modal logics over intuitionistic logic by Ono [27]. Ono shows that over an **S4**-type intuitionistic

[1] For relevant modal logics, see [9,35,36]. For an overview of modal logics based on **FDE** with a conditional from classical logic, see [22]. More recent developments include [6,18–20,34], and those with dynamic flavour include [33]. For an overview of **FDE**, see [26].

Special Issue: In Memory of Nuel Belnap (1930–2024)
Edited by **Thomas Müller, Tomasz Placek, and Heinrich Wansing**.

Studia Logica
https://doi.org/10.1007/s11225-026-10255-8
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Published online: 10 November 2026

modal logic, four different classically equivalent axioms give rise to distinct logics. These logics have natural frame conditions, and many interesting logical features. While intuitionistic logic does not have direct connections to **FDE**, we will see that many of the ideas from Ono's work will be applicable to the **FDE** context.

Second, there has been recent investigation into **S5**-type logics in the context of relevant logics by Standefer [40]. The starting point of that work is noting that there are three different presentations of **S5** in the setting of classical logic. One is in terms of the more or less standard axioms and rules, another uses a universal relations on Kripke frames, and the third uses equivalence relations on Kripke frames.[2] Standefer shows that these three different approaches to **S5** yield three different logics when adapted to the relevant logical setting. This work is salient to our interests, since **FDE** is an important fragment of all the standard relevant logics.

Note also that there are a number of different ways to present **FDE**. In this paper, we will focus on the so-called the Australian plan and the American plan. On the Australian plan, negation is interpreted using the so-called Routley star, $*$, so that a negation holds at the world w iff the negated formula fails at w^*, which may be distinct from w.[3] On the American plan, formulas can be assigned true and false together, at a single world, and negation toggles between them.[4] In the basic language of **FDE**, the two plans are equivalent.[5] This equivalence is not guaranteed under expansions of the language. The addition of a necessity operator will provide us with an opportunity to explore the differences between the two plans.[6]

Against this background, this paper will present the basics for the **FDE**-based modal logics. The paper will proceed as follows. In the next section, we will define the basic models for modal logics over **FDE** using the two plans and prove some basic features of the resulting **K**-like logics. Then in §3, we will investigate different options for **S5**-like logics in both plans. We show that there are three different logics that can be obtained. In §4, we present sequent systems for five of the modal logics and demonstrate cut elimination for two of them. In §5, we demonstrate soundness and completeness with

[2]See also [43] for more discussion of universal necessity in non-classical logics.

[3]For discussion and development of the Australian plan, see [3,4,7,32].

[4]For discussion and development of the American plan, see [1,2,5,25,31].

[5]See, for example, Theorems 8.7.17 and 8.7.18 of [28].

[6]The modal expansions based on the Australian plan can be seen as addressing the question posed by Sergei Odintsov and Heinrich Wansing in [23, 54] concerning the self-extensional **FDE**-based modal logics.

respect to appropriate classes of models for all of our sequent systems. Then, we provide a summary of our results, discuss some of the upshots of our work, and close with some open questions in §6.

2. Semantics (I): K-Like Systems

The language $\mathcal{L}$ consists of the set $\{\sim, \wedge, \vee, \Box\}$ of propositional connectives and a countable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by Form the set of formulas defined as usual in $\mathcal{L}$. We denote a formula of $\mathcal{L}$ by A, B, C, etc. and a set of formulas of $\mathcal{L}$ by Γ, Δ, Σ, etc.

Let us begin with the American plan. The following semantics is due to Graham Priest, discussed in [28, Chapter 11a]. The system is called $\mathbf{K_{FDE}}$ by Priest, but we refer to it under a different name for reasons that will become clear below. Alternatively, the following semantics is obtained by removing the conditional and the falsum constant from the semantics for the modal logic $\mathbf{BK}^{\Box}$ of Odintsov and Wansing, introduced in [22] which is equivalent to **BK** introduced in [21] if $\Diamond A$ is treated as an abbreviation of $\sim\Box\sim A$.

Definition 1. A $\mathbf{K_{4FDE}}$-model for the language $\mathcal{L}$ is a triple $\langle W, R, V\rangle$, where W is a non-empty set (of states); R is a binary relation on W; and $V : W \times \mathsf{Prop} \longrightarrow \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ is an assignment of truth values to state-variable pairs. Valuations V are then extended to interpretations I to state-formula pairs by the following conditions:[7]

- $I(w, p) = V(w, p)$,
- $1 \in I(w, \sim A)$ iff $0 \in I(w, A)$,
- $0 \in I(w, \sim A)$ iff $1 \in I(w, A)$,
- $1 \in I(w, A \wedge B)$ iff $1 \in I(w, A)$ and $1 \in I(w, B)$,
- $0 \in I(w, A \wedge B)$ iff $0 \in I(w, A)$ or $0 \in I(w, B)$,
- $1 \in I(w, A \vee B)$ iff $1 \in I(w, A)$ or $1 \in I(w, B)$,
- $0 \in I(w, A \vee B)$ iff $0 \in I(w, A)$ and $0 \in I(w, B)$,
- $1 \in I(w, \Box A)$ iff for all $x \in W$: (wRx only if $1 \in I(x, A)$),
- $0 \in I(w, \Box A)$ iff for some $x \in W$: (wRx and $0 \in I(x, A)$).

[7] As should be clear from the definition, we are using the Set-Fmla framework. See [14, 102ff.] for more on logical frameworks.

Finally, the semantic consequence is defined as follows: $\Gamma \models_{\mathbf{K_{4FDE}}} A$ iff for all $\mathbf{K}$-models $\langle W, R, I\rangle$, and for all $w \in W$: $1 \in I(w, A)$ if $1 \in I(w, B)$ for all $B \in \Gamma$.

We now turn to the case based on the Australian plan.

Definition 2. A $\mathbf{K_{*FDE}}$-model for the language $\mathcal{L}$ is a quadruple $\langle W, *, R, V\rangle$, where W is a non-empty set (of states); $* : W \longrightarrow W$ with $w^{**} = w$ for all $w \in W$; R is a binary relation on W; and $V : W \times \mathsf{Prop} \longrightarrow \{0, 1\}$ is an assignment of truth values to state-variable pairs. Valuations V are then extended to interpretations I to state-formula pairs by the following conditions:

- $I(w, p) = V(w, p)$,
- $I(w, \sim A) = 1$ iff $I(w^*, A) \neq 1$,
- $I(w, A \wedge B) = 1$ iff $I(w, A) = 1$ and $I(w, B) = 1$,
- $I(w, A \vee B) = 1$ iff $I(w, A) = 1$ or $I(w, B) = 1$,
- $I(w, \Box A) = 1$ iff for all $x \in W$: (wRx only if $I(x, A) = 1$),

Finally, the semantic consequence is now defined as follows: $\Gamma \models_{\mathbf{K_{*FDE}}} A$ iff for all $\mathbf{K_{*FDE}}$-models $\langle W, R, *, V\rangle$, and for all $w \in W$: $I(w, A) = 1$ if $I(w, B) = 1$ for all $B \in \Gamma$.

Remark 3. Note that we obtain the following for the truth condition for $\sim\Box\sim$:

$$\begin{aligned} I(w, \sim\Box\sim A) = 1 \text{ iff } & I(w^*, \Box\sim A) = 0 \\ & \text{iff for some } x_0 \in W : (w^* R x_0 \text{ and } I(x_0, \sim A) \neq 1) \\ & \text{iff for some } x_0 \in W : (w^* R x_0 \text{ and } I(x_0^*, A) = 1) \\ & \text{iff for some } x_1 \in W : (w^* R x_1^* \text{ and } I(x_1, A) = 1) \end{aligned}$$

Again, for a reason that will be clear in a moment, we introduce another semantics obtained by adding one condition.

Definition 4. A $\mathbf{K^+_{*FDE}}$-model for the language $\mathcal{L}$ is a $\mathbf{K_{*FDE}}$-model $\langle W, *, R, V\rangle$, obeying the condition

- (R^*) for all $x, y \in W$, xRy iff $x^* R y^*$.

We denote the semantic consequence relation as $\models_{\mathbf{K^+_{*FDE}}}$.

We note that, given that $x^{**} = x$, (R^*) can be simplified to a single conditional, the left-to-right direction. Suppose that $x^* R y^*$, so, by the left-to-right direction of (R^*), $x^{**} R y^{**}$, whence xRy.

REMARK 5. Given the (R^*) condition, we obtain the following for the truth condition for $\sim\Box\sim$, our defined $\Diamond$:

$$I(w, \sim\Box\sim A) = 1 \text{ iff for some } x_1 \in W : (wRx_1 \text{ and } I(x_1, A) = 1).$$

This matches the truth condition in classical Kripke models. Given (R^*), there is then a close connection between the defined $\Diamond$ and $\Box$.

The role of (R^*) is to force $\Box$ and $\Diamond$ to be interpreted on the same accessibility relation. Let us, for the purposes of this remark, add a primitive $\Diamond$ to our language and expand the $\mathbf{K}_{*\mathbf{FDE}}$-models, $\langle W, R_\Box, V\rangle$ with a second relation, $R_\Diamond$, for interpreting the $\Diamond$. The expanded, basic $\mathbf{K}_{*\mathbf{FDE}}$-models would allow $R_\Box$, for interpreting necessity, and $R_\Diamond$, for interpreting possibility, to be independent, but typically we want a connection between the concepts of necessity and possibility. Adding a modified form of (R^*), namely $\forall x, y \in W, xR_\Diamond y$ iff $x^*R_\Box y^*$, ensures that the two modalities are interpreted on the same relation and are, in fact, duals. When there are two primitive modalities, with two relations, we can see the contributions of the two directions of the condition. The $xR_\Diamond y \Rightarrow x^*R_\Box y^*$ direction ensures $\Diamond A \models \sim\Box\sim A$ and $\Box A \models \sim\Diamond\sim A$. The $x^*R_\Box y^* \Rightarrow xR_\Diamond y$ direction ensures $\sim\Box\sim A \models \Diamond A$ and $\sim\Diamond\sim A \models \Box A$. Having made this point, we will, for the remainder of the paper, no longer consider a primitive possibility operator.[8]

Let us now turn to how these three consequence relations are related.

PROPOSITION 6. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{K}_{*\mathbf{FDE}}} A$ *only if* $\Gamma \models_{\mathbf{K}^+_{*\mathbf{FDE}}} A$. *However, the converse does not hold. Indeed,*

- $\Box(A \vee B) \models_{\mathbf{K}^+_{*\mathbf{FDE}}} \Box A \vee \sim\Box\sim B$,
- $\sim\Box\sim A \wedge \Box B \models_{\mathbf{K}^+_{*\mathbf{FDE}}} \sim\Box\sim(A \wedge B)$, *but*
- $\Box(p \vee q) \not\models_{\mathbf{K}_{*\mathbf{FDE}}} \Box p \vee \sim\Box\sim q$, *and*
- $\sim\Box\sim p \wedge \Box q \not\models_{\mathbf{K}_{*\mathbf{FDE}}} \sim\Box\sim(p \wedge q)$.

PROOF. For the first item, assume that there is a $\mathbf{K}^+_{*\mathbf{FDE}}$-model $\langle W, *, R, V\rangle$ and $w_0 \in W$ such that $I(w_0, \Box(A \vee B)) = 1$ and $I(w_0, \Box A \vee \sim\Box\sim B) = 0$. The latter condition is equivalent to:

- for some $w_1 \in W_0$, w_0Rw_1 and $I(w_1, A) = 0$ and
- for all $x \in W_0$, w_0Rx only if $I(x, B) = 0$.

These conditions imply that w_0Rw_1 and $I(w_1, A) = 0$, and $I(w_1, B) = 0$, but these contradict $I(w_0, \Box(A \vee B)) = 1$. The second item is similar.

[8]We would like to thank Lloyd Humberstone for discussion.

For the third item, consider the model with

- $W := \{w_0, w_1, w_2\}$;
- $w_0^* = w_1$, $w_1^* = w_0$, $w_2^* = w_2$;
- $R := \{\langle w, w\rangle \mid w \in W\} \cup \{\langle w_0, w_2\rangle, \langle w_2, w_0\rangle\}$;
- Define V such as p is true only at w_0 and q is true only at w_2.

Then, we have that $I(w_0, \Box(p \vee q) = 1)$ and $I(w_0, \Box p) = 0$. Since $I(w_0, q) = 0$, $I(w_1, {\sim}q) = 1$, and since w_1 is the only R-successor of w_1, $I(w_1, \Box{\sim}q) = 1$. Therefore, $I(w_0, {\sim}\Box{\sim}q) = 0$, and so $\Box(p \vee q) \not\models_{\mathbf{K_{*FDE}}} \Box p \vee {\sim}\Box{\sim}q$, as desired.

We leave verification of the fourth to the interested reader. ■

It is worth noting that the particular examples we have chosen in the preceding proposition appear elsewhere. They play an important role in Dunn's [8] work on positive modal logic.[9] The first also appears in work on relevant modal logic in conditional form, and it is sometimes called "modal confinement."[10]

While there is a gap between the logic of the $\mathbf{K_{*FDE}}$-models and that of the $\mathbf{K^+_{*FDE}}$-models, we can show that there is no such gap between the $\mathbf{K^+_{*FDE}}$-models and the $\mathbf{K_{4FDE}}$-models. We will do this by showing that from one type of model, we can generate an equivalent model of the other type.

Definition 7. Let $\langle W, R, V\rangle$ be a $\mathbf{K_{4FDE}}$-model. Then its *star-mate* is $\langle W', *, R', V'\rangle$, where

- $W' = W \times \{0, 1\}$
- $(w, 1)^* = (w, 0)$,
- $(w, 0)^* = (w, 1)$,
- $R' = \{((x, 1), (y, 1)) \in W' \times W' : xRy\} \cup \{((x, 0), (y, 0)) \in W' \times W' : xRy\}$,
- $V'((w, 1), p) = 1$ iff $1 \in V(w, p)$, and
- $V'((w, 0), p) = 1$ iff $0 \notin V(w, p)$.

Lemma 8. *Let $\langle W, R, V\rangle$ be a $\mathbf{K_{4FDE}}$-model. and $\langle W', *, R', V'\rangle$ its star-mate. For all $x, y \in W$, for $i, j \in \{0, 1\}$, $(x, i)R'(y, j)$ only if $i = j$.*

[9] These sequents are called the Dunn conditions by [30, 266]. For more discussion of positive modal logic, see [13].

[10] See, for example, [16].

PROOF. Suppose that $(x,i)R'(y,j)$ but $i \neq j$. Without loss of generality, we suppose $i = 1$ and $j = 0$. Then $(x,1)R'(y,0)$, but this contradicts the definition of R'. Therefore $i = j$. ■

LEMMA 9. *Let* $\langle W, R, V\rangle$ *be a* $\mathbf{K_{4FDE}}$*-model. and* $\langle W', *, R', V'\rangle$ *its star-mate. The star-mate is a* $\mathbf{K^{+}_{*FDE}}$*-model.*

PROOF. Let $\langle W, R, V\rangle$ be a $\mathbf{K_{4FDE}}$-model. The only conditions on $\langle W \times \{0,1\}, *, R', V'\rangle$ being a model that needs to be checked is that for $x \in W'$, $x^{**} = x$ and if $xR'y$, then $x^*R'y^*$.

First, we show that $x^{**} = x$. From the definition, x is either $(w,1)$ or $(w,0)$, for some $w \in W$. Suppose the former. It then follows that $(w,1)^{**} = (w,0)^* = (w,1)$. The other case is similar.

Suppose that for some $xR'y$. So, for some $w, z \in W$, $x = (w,i)$ and $y = (z,j)$, for $i,j \in \{0,1\}$. By lemma 8, $i = j$, so without loss of generality we suppose that $i = j = 1$. Then, $(w,1)^* = (w,0)$ and $(z,1)^* = (z,0)$. Since $(w,1)R'(z,1)$, it follows that wRz, whence $(w,0)R'(z,0)$, which is $(w,1)^*R'(z,1)^*$, as desired.

That V' is well-defined is immediate from the definition. ■

LEMMA 10. *Let* $\langle W, R, V\rangle$ *be a* $\mathbf{K_{4FDE}}$*-model. and* $\langle W', *, R', V'\rangle$ *its star-mate. For all formulas* A, $1 \in I(w,A)$ *iff* $I'((w,1),A) = 1$ *and* $0 \in I(w,A)$ *iff* $I'((w,1)^*,A) = 0$.

PROOF. By induction on A. The base case is immediate from the definition of V'.

The cases where A is either $B \wedge C$ or $B \vee C$ are immediate from the inductive hypothesis.

Suppose A is $\sim B$.

$$\begin{aligned} 1 \in I(w, \sim B) &\text{ iff } 0 \in I(w,B) \\ &\text{iff } I'((w,1)^*, B) = 0 \\ &\text{iff } I'((w,1), \sim B) = 1 \end{aligned}$$

The second line is justified by the inductive hypothesis, and the other lines are justified by the definitions of the models. The case where $0 \in I(w, \sim B)$ is similar.

Next, let A be $\Box B$. Suppose $1 \in I(w, \Box B)$. This is true iff for all $u \in W$: if wRu, then $1 \in I(u,B)$. By the inductive hypothesis, $I'((u,1),B) = 1$. Suppose $(w,1)R'x$. By lemma 8, it follows that for some $z \in W$, $x = (z,1)$. By definition of R', $(w,1)R'(z,1)$ only if wRz, so $I'((z,1),B) = 1$. Therefore, $I'((w,1), \Box B) = 1$.

Next, suppose $I'((w,1),\Box B) = 1$. Then for all $(x,1) \in W'$ such that $(w,1)R'(x,1)$, it follows that $I'((x,1),B) = 1$. Suppose wRy. This implies $(w,1)R'(y,1)$, so $I'((y,1),B) = 1$. By the inductive hypothesis, $1 \in I(y,B)$, whence $1 \in I(w,\Box B)$.

Next, suppose $0 \in I(w,\Box B)$. So, there is some $x \in W$ such that wRx and $0 \in I(x,B)$. By the inductive hypothesis, $I'((x,0),B) = 0$. Since wRx, it follows that $(w,0)R'(x,0)$, so $I'((w,0),\Box B) = 0$, as desired.

Finally, suppose that $I'((w,0),\Box B) = 0$. This implies for some $x \in W$, $(w,0)R'(x,0)$ and $I'((x,0),B) = 0$. By the inductive hypothesis, $0 \in I(x,B)$. Since $(w,0)R'(x,0)$, wRx, whence $0 \in I(w,\Box B)$.

■

Just as we can generate a $\mathbf{K^+_{*FDE}}$-model from a $\mathbf{K_{4FDE}}$-model, we can do the converse.

Definition 11. Let $\langle W, R, *, V\rangle$ be a $\mathbf{K^+_{*FDE}}$-model. Its *4-mate* is $\langle W, R, V'\rangle$, where

- $1 \in V'(w,p)$ iff $V(w,p) = 1$, and
- $0 \in V'(w,p)$ iff $V(w^*,p) = 0$.

Lemma 12. *Let $\langle W, R, *, V\rangle$ be a $\mathbf{K^+_{*FDE}}$-model and $\langle W, R, V'\rangle$ its 4-mate. Then $\langle W, R, V'\rangle$ is a $\mathbf{K_{4FDE}}$-model.*

Proof. The only thing to check is that V' is well-defined. It is, since V is well-defined. ■

Lemma 13. *Let $\langle W, R, *, V\rangle$ be a $\mathbf{K^+_{*FDE}}$-model and $\langle W, R, V'\rangle$ its 4-mate. Then for all $w \in W$ and all formulas A, both*

- $1 \in I'(w,A)$ *iff* $I(w,A) = 1$, *and*
- $0 \in I'(w,A)$ *iff* $I(w^*,A) = 0$.

Proof. The proof is by induction on the structure of A.

The base cases are immediate from the definitions.

The inductive cases where A is either $B \wedge C$ or $B \vee C$ are immediate from the inductive hypothesis.

Suppose A is $\sim B$.

$$\begin{aligned} 1 \in I'(w,\sim B) &\text{ iff } 0 \in I(w,B) \\ &\text{iff } I(w^*,B) = 0 \\ &\text{iff } I(w,\sim B) = 1 \end{aligned}$$

The second line is justified by the inductive hypothesis. The case where $0 \in I'(w, {\sim}B)$ is similar.

Next, suppose that A is $\Box B$. Suppose $1 \in I'(w, \Box B)$. Then for all $u \in W$, if wRu, $1 \in I'(u, B)$. By the inductive hypothesis, $I(u, B) = 1$, but this is the case iff $I(w, \Box B) = 1$.

Suppose $0 \in I'(w, \Box B)$. Then for some $u \in W$, wRu and $0 \in I'(u, B)$. By the inductive hypothesis, $I(u^*, B) = 0$. Since wRu, w^*Ru^*, so for some $y \in W$, w^*Ry and $I(y, B) = 0$. This is the case iff $I(w^*, \Box B) = 0$. ■

Given the preceding results, we can conclude the following proposition.

Proposition 14. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{K_{4FDE}}} A$ *iff* $\Gamma \models_{\mathbf{K^+_{*FDE}}} A$.

Proof. By lemmas 10 and 13, a countermodel from one class generates a countermodel from the other. ■

These results show already that the different semantic plans for negation in **FDE** have some subtle repercussions, justifying our use of multiple labels for the different semantic consequence relations.

3. Semantics (II): S5-Like Systems

We now turn to the stronger end of the range of systems under consideration.

Definition 15. We introduce the following six models for the language $\mathcal{L}$.

- $\mathbf{K}\rho\sigma\tau_{\mathbf{4FDE}}$-model: obtained from $\mathbf{K_{4FDE}}$-model by requiring reflexivity, symmetry and transitivity for R.
- $\mathbf{K}\upsilon_{\mathbf{4FDE}}$-model: : obtained from $\mathbf{K_{4FDE}}$-model by requiring universality for R.
- $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$-model: obtained from $\mathbf{K_{*FDE}}$-model by requiring reflexivity, symmetry and transitivity for R.
- $\mathbf{K}\upsilon_{*\mathbf{FDE}}$-model: obtained from $\mathbf{K_{*FDE}}$-model by requiring universality for R.
- $\mathbf{K}\rho\sigma\tau^+_{*\mathbf{FDE}}$-model: obtained from $\mathbf{K^+_{*FDE}}$-model by requiring reflexivity, symmetry and transitivity for R.
- $\mathbf{K}\upsilon^+_{*\mathbf{FDE}}$-model: obtained from $\mathbf{K^+_{*FDE}}$-model by requiring universality for R.

We denote the associated semantic consequence relations in the expected manner.

The rest of this section will clarify the relation of the above six consequence relations.

PROPOSITION 16. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{K}\rho\sigma\tau_{\mathbf{4FDE}}} A$ *iff* $\Gamma \models_{\mathbf{K}\upsilon_{\mathbf{4FDE}}} A$.

PROOF. The left-to-right direction is immediate since universal relations are reflexive, symmetric, and transitive.

For the other way around, suppose that $\Gamma \not\models_{\mathbf{K}\rho\sigma\tau_{\mathbf{4FDE}}} A$. So, for some model, there is $w \in W$ such that $1 \in I(B, w)$, for all $B \in \Gamma$, but $1 \notin I(A, w)$. Let $W' = \{u \in W : wRu\}$ and let $R' = \{(x, y) \in W' \times W' : xRy\}$. Let I' be defined as $I'(A, w) = I(A, w)$, for $w \in W'$. Taking these together gives us a model, $\langle W', R', I' \rangle$, and we will now show that R' is the universal relation on $W' \times W'$.

Let $x, y \in W'$. It follows that wRx and wRy. Since R is an equivalence relation, xRy. Therefore R' is the universal relation on $W' \times W'$.

Next, by induction on the structure of formulas, $I'(A, w) = I(A, w)$, for all $w \in W'$. It follows that $1 \in I'(B, w)$, for all $B \in \Gamma$ but $1 \notin I'(A, w)$, so $\Gamma \not\models_{\mathbf{K}\upsilon_{\mathbf{4FDE}}} A$ ∎

To facilitate some of the comparisons, we will show that the star-mate construction preserves the relevant conditions on the accessibility relations, in particular being reflexive, symmetric, and transitive. It is immediate that the 4-mate construction preserves the conditions on the accessibility relation R, since the construction does not change the accessibility relation of the model.

LEMMA 17. *Let* $\langle W, R, V \rangle$ *be a* $\mathbf{K_{4FDE}}$*-model and* $\langle W', *, R', V' \rangle$ *its star-mate. Then,*

- *if* R *is reflexive, so is* R'*,*
- *if* R *is symmetric, so is* R'*, and*
- *if* R *is transitive, so is* R'*.*

PROOF. Suppose R is reflexive. Let $w \in W$. Since R is reflexive, wRw, so $(w, 1)R'(w, 1)$ and $(w, 0)R'(w, 0)$. Therefore, R' is reflexive.

Suppose R is symmetric. Suppose for $w, x \in W$, $(w, i)R'(x, i)$, for $i \in \{0, 1\}$. It follows that wRx, so by symmetry, xRw. Therefore, $(x, i)R'(w, i)$.

Suppose R is transitive. Suppose for $x, y, z \in W$, $(x, i)R'(y, i)$ and $(y, i)R'(z, i)$, for $i \in \{0, 1\}$. It follows that xRy and yRz, so by transitivity, xRz. Therefore, $(x, i)R'(z, i)$. ∎

Universality is not preserved by the star-mate construction. Take the $\mathbf{K_{4FDE}}$-model with $W = \{w\}$, $R = \{(w, w)\}$, and $V(w, p) = \{0, 1\}$, for

$p \in \mathsf{Prop}$. The star-mate generated will have $W' = \{(w,0),(w,1)\}$ and $R' = \{((w,0),(w,0)),((w,1),(w,1))\}$. The relation thus obtained is not in general universal, though it is at least guaranteed to be an equivalence relation.

PROPOSITION 18. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{K}\rho\sigma\tau_{\mathbf{4FDE}}} A$ *iff* $\Gamma \models_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} A$.

PROOF. If $\Gamma \not\models_{\mathbf{K}\rho\sigma\tau_{\mathbf{4FDE}}} A$, then there is a $\mathbf{K}\rho\sigma\tau_{\mathbf{4FDE}}$-countermodel. Its star-mate is $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$-countermodel, by lemmas 10 and 17. If $\Gamma \not\models_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} A$, then there is a $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$-countermodel. Its 4-mate has the same accessibility relation, so it is a $\mathbf{K}\rho\sigma\tau_{\mathbf{4FDE}}$-countermodel by lemma 13. ■

PROPOSITION 19. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} A$ *only if* $\Gamma \models_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} A$. *However, the converse does not hold. Indeed,*

- $\Box(A \vee B) \models_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} \Box A \vee {\sim}\Box{\sim}B$,
- ${\sim}\Box{\sim}A \wedge \Box B \models_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} {\sim}\Box{\sim}(A \wedge B)$, *but*
- $\Box(p \vee q) \not\models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box p \vee {\sim}\Box{\sim}q$, *and*
- ${\sim}\Box{\sim}p \wedge \Box q \not\models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} {\sim}\Box{\sim}(p \wedge q)$.

PROOF. The implication is immediate.

The first bullet point holds from the proof of the corresponding claim in the proof of proposition 6. The second is similar.

For the third bullet point, we can use the model from the proof of proposition 6. We leave verification of the fourth bullet point to the interested reader. ■

PROPOSITION 20. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{K}\upsilon_{*\mathbf{FDE}}} A$ *iff* $\Gamma \models_{\mathbf{K}\upsilon^{+}_{*\mathbf{FDE}}} A$.

PROOF. The left-to-right direction is obvious. For the other way around, if the binary relation is universal, then the (R^*) condition is also satisfied. ■

For the next result, it will be useful to prove a small lemma.

LEMMA 21. $\langle W, R, *, V\rangle$ *be a* $\mathbf{K}\upsilon_{*\mathbf{FDE}}$*-model. For any* $w \in W$, *if* $I(w,\Box A) = 1$, *then for all* $x \in W$, $I(x,\Box A) = 1$, *and if* $I(w,\Box A) = 0$, *then for all* $x \in W$, $I(x,\Box A) = 0$.

PROOF. Suppose $I(w,\Box A) = 1$. It follows that for all $y \in W$, $I(y,A) = 1$. Therefore, for arbitrary $x \in W$, $I(x,\Box A) = 1$.

Next, suppose $I(w,\Box A) = 0$. For some $y \in W$, $I(y,A) = 0$, so for arbitrary $x \in W$, $I(x,\Box A) = 0$. ■

In the terminology of Standefer [38,39], many $\Box$-formulas are *ubiquitous* in the class of $\mathbf{K}\upsilon_{*\mathbf{FDE}}$-models. This fact will have many consequences, to be explored below. The first among these consequences we will note is that $\mathbf{K}\upsilon_{*\mathbf{FDE}}$ outstrips $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$.

PROPOSITION 22. *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} A$ *only if* $\Gamma \models_{\mathbf{K}\upsilon_{*\mathbf{FDE}}} A$. *However, the converse does not hold. Indeed,*

- $\models_{\mathbf{K}\upsilon_{*\mathbf{FDE}}} \Box A \vee \Box{\sim}\Box A$, *but*
- $\not\models_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} \Box p \vee \Box{\sim}\Box p$.

PROOF. For the first item, let $\langle W, R, *, V\rangle$ be a $\mathbf{K}\upsilon_{*\mathbf{FDE}}$-model with $w \in W$. Suppose $I(w, \Box A) = 0$. By lemma 21, $I(x, \Box A) = 0$, for all $x \in W$. So, for all $y \in W$, $I(y^*, \Box A) = 0$. Therefore, for all $y \in W$ $I(y, {\sim}\Box A) = 1$. Therefore, $I(w, \Box{\sim}\Box A) = 1$.

For the second item, let us consider the following model.

- $W = \{w_0, w_1\}$.
- $w_0^* = w_1$, $w_1^* = w_0$.
- $w_0 R w_0, w_1 R w_1$.
- $V(w_1, p) = 1$, $V(w_0, p) = 0$.

This is a $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$-model. It is clear that $I(w_0, \Box p) = 0$. Further, $I(w_1, \Box p) = 1$, so $I(w_0, {\sim}\Box p) = 0$, whence $I(w_0, \Box{\sim}\Box p) = 0$, as desired. ■

The formula $\Box A \vee \Box{\sim}\Box A$ is the axiom A_4 of Ono [27]. It follows from Ono's work that, in the context of Kripke models for intuitionistic logic, this axiom can be used to axiomatize the logic of frames whose modal accessibility relation is universal.

As the results of this section show, there are at least three distinct systems that can be regarded as **S5**-type logics over **FDE**, and the relations among them can be summarized by the following table:

$$\models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \subsetneq \models_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} = \models_{\mathbf{K}\rho\sigma\tau_{\mathbf{4FDE}}} = \models_{\mathbf{K}\upsilon_{\mathbf{4FDE}}} \subsetneq \models_{\mathbf{K}\upsilon_{*\mathbf{FDE}}} = \models_{\mathbf{K}\upsilon^{+}_{*\mathbf{FDE}}}$$

With the relations between the systems in view, we can see that in some cases, the Australian plan systems are weaker than their American plan relatives. The key to this perhaps surprising fact is the interplay between the star and accessibility relation. For the logics whose models need not have universal accessibility relations, such as $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$, the star need not respect the accessibility relation, in the sense that one can have wRv but not w^*Rv^*, which allows for the counterexample ensuring $\Box(p \vee q) \not\models_{\mathbf{K}_{*\mathbf{FDE}}} \Box p \vee {\sim}\Box{\sim} q$. With the American plan models, the interpretation of negation does not

interact with the accessibility relation. Inspection of the proof of Lemma 9 reveals that generating the star-mate of an American plan model results in the star respecting the accessibility relation in the constructed model. Thus, we see that that the interplay of the Routley star and the modal accessibility relation is key to the divergence of the systems.

For logics whose models all have universal accessibility relations, such as $\mathbf{K}\upsilon_{*\mathbf{FDE}}$, the star must respect the accessibility relation. The difference between the Australian and American plan models depends on the way negation is interpreted. In the Australian plan models, $\sim A$ holds at a point iff A fails at the point's star-mate. The evaluation of $\Box$ formulas requires looking at all points of the model, so if $\Box A$ holds somewhere in a model, A holds everywhere and so there is nowhere for A to fail. Consequently, there is nowhere at which $\sim A$ or $\sim\Box A$ holds. By contrast, the American plan models can have $\Box A$ be true at a point, while $\sim\Box A$ is also true at that point, because truth and falsity at a point do not depend upon failing to have a value at some point.

Before we proceed to the proof theory, we want to show that some, but not all, of the logics enjoy the variable-sharing property.

Definition 23. A logic $\mathbf{L}$ enjoys the variable-sharing property iff whenever $\Gamma \models_{\mathbf{L}} A$, then some atom occurs both in some $B \in \Gamma$ and in A.

Proposition 24. *The logic* $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$ *enjoys the variable-sharing property.*

Proof. Suppose that for $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} A$ and that there is no variable shared between Γ and A. Let $\langle W, R, *, V\rangle$ be such that $W = \{x, y\}$, xRx, yRy, $x^* = y$ and $y^* = x$. We define V as follows.

- If p occurs in Γ in the scope of an even number of negations, set $V(x, p) = 1$.
- If p occurs in Γ in the scope of an odd number of negations, set $V(y, p) = 0$.
- If q occurs in A in the scope of an even number of negations, set $V(x, q) = 0$.
- If q occurs in A in the scope of an even number of negations, set $V(y, q) = 1$.

From the definition of R, it follows that for all formulas B and $u \in W$, $I(u, \Box B) = I(u, B)$.

We can show that if B contains only atoms occurring in Γ, then $I(x, B) = 1$ and $I(y, B) = 0$, and if B contains only atoms occurring in A, then

$I(x, B) = 0$ and $I(y, B) = 1$ Both of these claims are proved by a straightforward induction on the construction of B.

It follows that $I(x, B) = 1$, for $B \in \Gamma$ while $I(x, A) = 0$. Therefore, $\Gamma \not\models_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} A$, as desired. ■

Lemma 25. $\Box p, {\sim}\Box p \models_{\mathbf{K}\upsilon_{*\mathbf{FDE}}} q$

Proof. Let $\langle W, R, *, V\rangle$ be a $\mathbf{K}\upsilon_{*\mathbf{FDE}}$-model and $w \in W$. Suppose $I(w, \Box p) = 1$ and $I(w, {\sim}\Box p) = 1$. The latter implies $I(w^*, \Box p) = 0$, so by lemma 21, $I(w, p) = 0$. This implies $I(w, \Box p) = 0$, which contradicts an assumption. Therefore, there is no such model with $I(w, \Box p) = 1$ and $I(w, {\sim}\Box p) = 1$. ■

Proposition 26. $\mathbf{K}\upsilon_{*\mathbf{FDE}}$ *does not enjoy the variable-sharing property*

Proof. This follows from the preceding lemma. ■

The logic $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$ enjoys the variable-sharing property, and so, therefore, do all of its sublogics. Its strengthening $\mathbf{K}\upsilon_{*\mathbf{FDE}}$, however, does not enjoy the variable-sharing property, so there is a substantial transition in going from the star-models with an equivalence relation to the star-models with a universal relation.[11]

Let us say that a consequence relation $\models$ is *purely inferential* when there is no formula A such that $\emptyset \models A$.[12] All of our logics that enjoy the variable-sharing property are purely inferential. To see this, let $\models$ the consequence relation of one of those logics and suppose that we have $\models A$, for some formula A. It would then follow that $p \models A$, for some atom p not occurring in A, contradicting the assumption that $\models$ has the variable-sharing property. So, a logic having the variable-sharing property ensures that its consequence relation is purely inferential. While Lemma 25 tells us that $\mathbf{K}\upsilon_{*\mathbf{FDE}}$ lacks the variable-sharing property, we cannot infer from that $\models_{\upsilon_{*\mathbf{FDE}}}$ is not purely inferential. We can, however, prove a variant the lemma, using analogous reasoning, that establishes that ${\sim}q \models_{\upsilon_{*\mathbf{FDE}}} {\sim}\Box p \vee \Box p$. Consideration of the argument reveals that we can freely drop the ${\sim}q$ on the left, establishing that the consequence relation is not purely inferential.[13]

[11] Standefer [42] discusses the significance of the variable-sharing property. On his view, $\mathbf{K}\upsilon_{*\mathbf{FDE}}$ would not count as a relevant logic, while $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$ would.

[12] The term 'purely inferential' is due to Wójcicki [48, 41]. In the terminology of Humberstone [14, 204-205], such consequence relations are atheorematic.

[13] We would like to thank Lloyd Humberstone for discussion of this point.

4. Proof Theory

In this section, we will present some proof systems for our logics. To motivate some of our choices we will begin by noting some valid arguments and rules for the different logics. We will, below, adopt the notation $\#\Gamma = \{\#A : A \in \Gamma\}$, where $\#$ is any sequence of one-place connectives.

PROPOSITION 27. *The following hold for* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$.

- $\Box A, \Box B \models_{\mathbf{K}_{*\mathbf{FDE}}} \Box(A \wedge B)$.
- *If* $\Gamma \models_{\mathbf{K}_{*\mathbf{FDE}}} A$, *then* $\Box\Gamma \models_{\mathbf{K}_{*\mathbf{FDE}}} \Box A$.
- $\Diamond(A \vee B) \models_{\mathbf{K}_{*\mathbf{FDE}}} \Diamond A \vee \Diamond B$.
- *If* $A \models_{\mathbf{K}_{*\mathbf{FDE}}} B$, *then* $\Diamond A \models_{\mathbf{K}_{*\mathbf{FDE}}} \Diamond B$.

Here $\Diamond$ *is an abbreviation for* $\sim\Box\sim$.

PROOF. For this proof, let $\langle W, R, *, V\rangle$ be a $\mathbf{K}_{*\mathbf{FDE}}$-model and $w \in W$. For the first, suppose $I(w, \Box A) = I(w, \Box B) = 1$. It follows that for $x \in W$, if wRx, then $I(x, A) = I(x, B) = 1$, so $I(w, \Box(A \wedge B)) = 1$.

For the second, suppose $\Gamma \models_{\mathbf{K}_{*\mathbf{FDE}}} A$ and $I(w, \Box B) = 1$, for $\Box B \in \Box\Gamma$. Let $x \in W$ be such that wRx. It follows that $I(x, B) = 1$, for $B \in \Gamma$. By the assumption, $I(x, A) = 1$, so $I(w, \Box A) = 1$.

The proof for the third is similar to that of the first, so we omit it.

For the fourth, suppose $A \models_{\mathbf{K}_{*\mathbf{FDE}}} B$ and let $\langle W, R, *, V\rangle$ be a $\mathbf{K}_{*\mathbf{FDE}}$-model and $w \in W$ such that $I(w, \Diamond A) = 1$. This is equivalent to $I(w^*, \Box\sim A) = 0$, which holds iff for some $x \in W$, w^*Rx and $I(x, \sim A) = 0$. That $I(x, \sim A) = 0$ is equivalent to $I(x^*, A) = 1$. From the initial assumption, we can conclude $I(x^*, B) = 1$. Reversing the equivalences gives $I(w, \Diamond B) = 1$, as desired. ■

All four of these bullet points remain true when any of the logics from the preceding two sections are used. We note that for the second bullet point, when $\Gamma = \emptyset$, the rule is the standard rule of necessitation from normal modal logics. This is worth noting given the connection between **FDE** and relevant logics, because in the setting of relevant modal logics, the rule of necessitation does not hold without additional conditions.[14]

We now turn to some validities that are particular to the class of $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$-models.

PROPOSITION 28. *The following hold for* $\{A, B\} \subseteq \mathsf{Form}$.

[14] See [9] for details.

- $\Box A \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} A$.
- $A \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Diamond A$.
- $\Box A \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box\Box A$.
- $\Diamond\Diamond A \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Diamond A$.
- $\Box(A \vee \Box B) \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box A \vee \Box B$.

Proof. We give proofs of the second, fourth, and fifth items. Let $\langle W, R, *, V\rangle$ be a $\mathbf{K}_{*\mathbf{FDE}}$-model and $w \in W$.

For the second, suppose $I(w, A) = 1$ and $I(w, {\sim}\Box{\sim}A) = 0$. Then $I(w^*, \Box{\sim}A) = 1$. Since R is reflexive, $I(w^*, {\sim}A) = 1$, so $I(w, A) = 0$, which contradicts our assumption.

For the fourth, suppose $I(w, \Diamond\Diamond A) = 1$ and $I(w, {\sim}\Box{\sim}A) = 0$. It follows that $I(w^*, \Box{\sim}A) = 1$, so for all $x \in W$, if w^*Rx, then $I(x, {\sim}A) = 1$. By assumption, $I(w^*, \Box{\sim}\Diamond A) = 0$, so for some $y \in W$, wRy and $I(y, {\sim}{\sim}\Box{\sim}A) = 0$. This implies $I(y, \Box{\sim}A) = 0$, so for some $z \in W$, yRz and $I(z, {\sim}A) = 0$, so $I(z^*, A) = 1$. As wRy and yRz, by transitivity, wRz, so $I(z, {\sim}A) = 1$, whence $I(z^*, A) = 0$, which is a contradiction.

For the fifth, suppose that $I(w, \Box(A \vee \Box B)) = 1$. It follows that for all $x \in W$, if wRx, then $I(x, A \vee \Box B) = 1$. This is equivalent to $I(x, A) = 1$ or $I(x, \Box B) = 1$. Suppose that $I(w, \Box A) = 0$ and $I(w, \Box B) = 0$. So, for some $y \in W$, wRy and $I(y, A) = 0$ and for some $z \in W$, wRz and $I(z, B) = 0$. Since $I(w, \Box(A \vee \Box B)) = 1$ and wRy, $I(y, A \vee \Box B) = 1$, so $I(y, \Box B) = 1$. By symmetry and transitivity, yRz, so $I(z, B) = 1$, which is a contradiction. ■

The fifth item is a form of the axiom A_3 of Ono [27]. A conditional form of it plays an important role in the completeness proof of Standefer and Mares [44], who call it (M5).

Proposition 29. *The following hold for* $A \in \mathsf{Form}$.

- $A \models_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} \Box\Diamond A$.
- $\Diamond A \models_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} \Box\Diamond A$.
- $p \not\models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box\Diamond p$.
- $\Diamond p \not\models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box\Diamond p$.

Proof. As Remark 5 shows, in $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$, $\Diamond$ $(= {\sim}\Box{\sim})$ obeys the same truth condition as in modal logics on classical logic. So the validity of the first and second items can be proved in the same way as in classical modal logic. Indeed, the symmetry of R suffices for the first item, and the second item holds under the symmetry and transitivity of R.

For the countermodel for the third item, let the model be

- $W = \{x, y, z\}$,
- xRx, yRy, zRz, xRy, yRx,
- $x^* = x$, $y^* = z$, $z^* = y$, and
- $V(x, p) = 1$, and $V(z, p) = 1$.

Then, $I(z, \sim p) = 1$, so $I(z, \Box\sim p) = 1$. This implies $I(z^*, \sim\Box\sim p) = 0$. Since $z^* = y$ and xRy, $I(x, \Box\sim\Box\sim p) = 0$.

For the fourth item, assume $\Diamond p \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box\Diamond p$. Since $p \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Diamond p$, from the transitivity of consequence, we have $p \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box\Diamond p$, which contradicts the invalidity demonstrated in the previous item. Therefore, $\Diamond p \not\models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box\Diamond p$. ■

Next, we will proceed to the sequent systems for $\mathbf{K}_{*\mathbf{FDE}}$, $\mathbf{K}^{+}_{*\mathbf{FDE}}$, and $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$. We begin with a fairly standard sequent system for **FDE**, to which we will add modal rules. We will use multi-succedent sequents, where each side of the turnstile is a finite multiset. We adopt the standard notational conventions for the area, e.g. Γ, Δ rather than $\Gamma \cup \Delta$ and Γ, A rather than $\Gamma \cup \{A\}$.

$$A \succ A \qquad \frac{\Gamma \succ \Delta, A \quad A, \Sigma \succ \Theta}{\Gamma, \Sigma \succ \Delta, \Theta}\ (\text{Cut})$$

$$\frac{A, A, \Gamma \succ \Delta}{A, \Gamma \succ \Delta}\ (KL) \qquad \frac{\Gamma \succ \Delta, A, A}{\Gamma \succ \Delta, A}\ (KR)$$

$$\frac{\Gamma \succ \Delta}{A, \Gamma \succ \Delta}\ (WL) \qquad \frac{\Gamma \succ \Delta}{\Gamma \succ \Delta, A}\ (WR)$$

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta}\ (\wedge L) \qquad \frac{\Gamma \succ \Delta, A \quad \Gamma \succ \Delta, B}{\Gamma \succ \Delta, A \wedge B}\ (\wedge R)$$

$$\frac{\Gamma, A \succ \Delta \quad \Gamma, B \succ \Delta}{\Gamma, A \vee B \succ \Delta}\ (\vee L) \qquad \frac{\Gamma \succ \Delta, A, B}{\Gamma \succ \Delta, A \vee B}\ (\vee R)$$

$$\frac{A, \Gamma \succ \Delta}{\sim\sim A, \Gamma \succ \Delta}\ (\sim\sim L) \qquad \frac{\Gamma \succ \Delta, A}{\Gamma \succ \Delta, \sim\sim A}\ (\sim\sim R)$$

$$\frac{\Gamma, \sim A, \sim B \succ \Delta}{\Gamma, \sim(A \vee B) \succ \Delta}\ (\sim \vee L) \qquad \frac{\Gamma \succ \Delta, \sim A \quad \Gamma \succ \Delta, \sim B}{\Gamma \succ \Delta, \sim(A \vee B)}\ (\sim \vee R)$$

$$\frac{\Gamma, \sim A \succ \Delta \quad \Gamma, \sim B \succ \Delta}{\Gamma, \sim(A \wedge B) \succ \Delta}\ (\sim \wedge L) \qquad \frac{\Gamma \succ \Delta, \sim A, \sim B}{\Gamma \succ \Delta, \sim(A \wedge B)}\ (\sim \wedge R)$$

To the non-modal rules, we add modal rules. For the system $\mathbf{GK}_{*\mathbf{FDE}}$ we add the following.

$$\frac{\Gamma \succ A}{\Box\Gamma \succ \Box A}\ (\Box) \qquad \frac{\sim A \succ \sim\Delta}{\sim\Box A \succ \sim\Box\Delta}\ (\sim\Box)$$

For the system $\mathbf{GK}^{+}_{*\mathbf{FDE}}$ we add the following.

$$\frac{\Gamma \succ A, \sim\Delta}{\Box\Gamma \succ \Box A, \sim\Box\Delta}\ (\Box^{+}) \qquad \frac{\Gamma, \sim A \succ \sim\Delta}{\Box\Gamma, \sim\Box A \succ \sim\Box\Delta}\ (\sim\Box^{+})$$

For the system $\mathbf{GK}\rho\sigma\tau_{*\mathbf{FDE}}$, we add instead the following rules.

$$\frac{\Gamma, A \succ \Delta}{\Gamma, \Box A \succ \Delta}\ (\Box L) \qquad \frac{\Box\Gamma \succ \Box\Delta, A}{\Box\Gamma \succ \Box\Delta, \Box A}\ (\Box R)$$

$$\frac{\sim\Box\Gamma, \sim A \succ \sim\Box\Delta}{\sim\Box\Gamma, \sim\Box A \succ \sim\Box\Delta}\ (\sim\Box L) \qquad \frac{\Gamma \succ \Delta, \sim A}{\Gamma \succ \Delta, \sim\Box A}\ (\sim\Box R)$$

Next, to obtain $\mathbf{GK}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$, we replace $(\Box R)$ and $(\sim\Box L)$ of $\mathbf{GK}\rho\sigma\tau_{*\mathbf{FDE}}$ by the following two rules: [15]

$$\frac{\sim\Box\Pi, \Box\Gamma \succ \Box\Delta, \sim\Box\Sigma, A}{\sim\Box\Pi, \Box\Gamma \succ \Box\Delta, \sim\Box\Sigma, \Box A}\ (\Box^{+}R) \qquad \frac{\Box\Pi, \sim\Box\Gamma, \sim A \succ \sim\Box\Delta, \Box\Sigma}{\Box\Pi, \sim\Box\Gamma, \sim\Box A \succ \sim\Box\Delta, \Box\Sigma}\ (\sim\Box^{+}L)$$

Finally, to obtain $\mathbf{GK}\upsilon_{*\mathbf{FDE}}$, we add the following rules to $\mathbf{GK}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$.[16]

$$\frac{\Gamma, \Box A \succ \Delta}{\Gamma, \succ\Delta, \sim\Box A}\ (\sim\Box^{\upsilon}R) \qquad \frac{\Gamma \succ \Delta, \Box A}{\sim\Box A, \Gamma \succ \Delta}\ (\sim\Box^{\upsilon}L)$$

For each of the sequent systems **GL** defined above, we will write $\mathbf{L} \vdash \Gamma \succ \Delta$ when there is a proof of $\Gamma \succ \Delta$ in **GL**.

It is worth pausing to comment on the rules. First, the modal rules that we have adopted are essentially those of Ohnishi and Matsumoto [24]. As should be clear, the $(\Box)$ and $(\sim\Box)$ rules are, respectively, special cases of the $(\Box^{+})$ and $(\sim\Box^{+})$ rules, which shows that $\mathbf{GK}^{+}_{*\mathbf{FDE}}$ contains $\mathbf{GK}_{*\mathbf{FDE}}$. The $(\Box R)$ and $(\sim\Box L)$ rules are derivable using $(\Box L)$, $(\Box R)$, $(\sim\Box L)$, and $(\sim\Box R)$,

[15] Alternatively, we add following axiom sequents to $\mathbf{GK}\rho\sigma\tau_{*\mathbf{FDE}}$.

- $\sim A \succ \Box\sim\Box A$
- $\sim\Box\sim\Box A \succ A$

[16] Alternatively, we add following axiom sequents to $\mathbf{GK}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$.

- $\sim\Box A, \Box A \succ$
- $\succ \Box A, \sim\Box A$

which shows that $\mathbf{GK}\rho\sigma\tau_{*\mathbf{FDE}}$ contains $\mathbf{GK}_{*\mathbf{FDE}}$. Similarly, the $(\Box^+)$ and $(\sim\Box^+)$ rules are derivable using the $(\Box L)$, $(\Box R^+)$, $(\sim\Box L^+)$, and $(\sim\Box R)$ rules, which shows that $\mathbf{GK}\rho\sigma\tau^+_{*\mathbf{FDE}}$ contains $\mathbf{GK}^+_{*\mathbf{FDE}}$. The rules $(\Box R)$ and $(\sim\Box L)$ are special cases of $(\Box R^+)$, $(\sim\Box L^+)$, so $\mathbf{GK}\rho\sigma\tau^+_{*\mathbf{FDE}}$ contains $\mathbf{GK}\rho\sigma\tau_{*\mathbf{FDE}}$. Finally, since we simply add rules to $\mathbf{GK}\rho\sigma\tau^+_{*\mathbf{FDE}}$ to obtain $\mathbf{GK}\upsilon_{*\mathbf{FDE}}$, the latter obviously contains the former. Thus, we have established proof-theoretic analogs of all the containments noted at the end of the previous section, although we have not proof-theoretically shown that those containments are proper.

Second, as is well known, when the **S5** rules are added to classical logic, cut is not eliminable. An example of a sequent whose derivation in the system of Ohnishi and Matsumoto requires the use of cut is $p \succ \Box\sim\Box\sim p$. One such derivation is the following.

$$\dfrac{\dfrac{\dfrac{\dfrac{p \succ p}{p, \sim p \succ}\,(c\sim L)}{p, \Box\sim p \succ}\,(\Box L)}{p, \succ \sim\Box\sim p}\,(c\sim R) \qquad \dfrac{\dfrac{\dfrac{\Box\sim p \succ \Box\sim p}{\succ \Box\sim p, \sim\Box\sim p}\,(c\sim R)}{\succ \Box\sim p, \Box\sim\Box\sim p}\,(\Box R)}{\sim\Box\sim p \succ \Box\sim\Box\sim p}\,(c\sim L)}{p \succ \Box\sim\Box\sim p}\,(\mathrm{Cut})$$

This derivation uses the classical negation rules, which are unavailable in our sequent system for $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$. Further, the endsequent of the derivation is invalid in $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$. Nonetheless, there is a sequent whose derivation requires cut.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box p \succ \Box p}{\Box p \succ \Box p, q}\,(WR) \qquad \dfrac{q \succ q}{q \succ \Box p, q}\,(WR)}{\Box p \vee q \succ \Box p, q}\,(\vee L)}{\Box(\Box p \vee q) \succ \Box p, q}\,(\Box L)}{\Box(\Box p \vee q) \succ \Box p, \Box q}\,(\Box R) \qquad \dfrac{p \succ p}{\Box p \succ p}\,(\Box L)}{\Box(\Box p \vee q) \succ p, \Box q}\,(\mathrm{Cut})$$

Third, we have adopted $\sim\Box$ rules rather than the $\Diamond$ rules of Ohnishi and Matsumoto, which we list below.

$$\dfrac{\Diamond\Gamma, A \succ \Diamond\Delta}{\Diamond\Gamma, \Diamond A \succ \Diamond\Delta}\,(\Diamond L) \qquad \dfrac{\Gamma \succ \Delta, A}{\Gamma \succ \Delta, \Diamond A}\,(\Diamond R)$$

These rules are, nonetheless, derivable, using the $\sim\sim$ rules followed by the $\sim\Box$ rules, since $\Diamond$ is defined as $\sim\Box\sim$.

Our final observation is that in our proof systems, the contraposition rule is admissible.

PROPOSITION 30. *For all* $\mathbf{L} \in \{\mathbf{K}_{*\mathbf{FDE}}, \mathbf{K}^{+}_{*\mathbf{FDE}}, \mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}, \mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}, \mathbf{K}\upsilon_{*\mathbf{FDE}}\}$, *if* $\mathbf{L} \vdash \Gamma \succ \Delta$, *then* $\mathbf{L} \vdash \sim\Delta \succ \sim\Gamma$.

PROOF. This is proved by induction on the construction of proofs. If an axiom is used, the contraposed form is also derivable. Every rule has a dual form that can be used in the dual proof to obtain the desired contraposed conclusion. We provide one case for $\mathbf{GK}_{*\mathbf{FDE}}$ to illustrate.

Suppose that the proof ends with the following application of $(\Box)$.

$$\dfrac{\begin{array}{c} D_0 \\ \Gamma \succ A \end{array}}{\Box\Gamma \succ \Box A}\ (\Box)$$

From the inductive hypothesis, there is a derivation D_1 of $\sim A \succ \sim\Gamma$, so we extend the proof with $(\sim\Box)$ to obtain the following.

$$\dfrac{\begin{array}{c} D_1 \\ \sim A \succ \sim\Gamma \end{array}}{\sim\Box A \succ \sim\Box\Gamma}\ (\sim\Box)$$

The other cases are similar. ∎

While $\mathbf{GK}\rho\sigma\tau_{*\mathbf{FDE}}$ and $\mathbf{GK}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$ do not enjoy cut elimination, $\mathbf{GK}_{*\mathbf{FDE}}$ and $\mathbf{GK}^{+}_{*\mathbf{FDE}}$ both do, which we will now prove.

THEOREM 1. *The cut rule is eliminable in* $\mathbf{GK}_{*\mathbf{FDE}}$.

PROOF. We prove that the following stronger version of cut rule is eliminable:

$$\dfrac{\Gamma \succ \Delta, A^m \qquad A^n, \Sigma \succ \Theta}{\Gamma, \Sigma \succ \Delta, \Theta}\ (\text{Mix})$$

Let the *length* of a formula be the number of symbols in the formula and the *height* of a deduction be the number of nodes in the maximal branch. The *level* of an instance of Mix in a deduction is the sum of the heights of the subductions of its premises. It is sufficient to argue that for any deduction whose only mix is the last rule can be transformed into a deduction without any mix. The proof proceeds by induction on the length of the cut formula A, and subinduction on the level of the mix. Note that for formulas $\sim(A \vee B)$, $\sim(A \wedge B)$, and $\sim\Box A$, $\sim A$ and $\sim B$ have fewer symbols. The analysis is according to the rules that derive the premises of the mix. For each case, the deduction under discussion can be transformed into either a deduction without a mix or one with several mixes, each of which either has a cut formula of smaller length or is one of the topmost mixes at a lower level. Then, by the induction hypothesis, the resulting deduction can be further transformed into a deduction without any mix. For most of the cases

the transformation is standard. We demonstrate modal cases to illustrate. Consider the following deduction D:

We focus on the case where the modal rules are used to obtain both premises.

Consider the following deduction D:

$$\dfrac{\dfrac{\begin{matrix}D_0\\ \Gamma \succ A\end{matrix}}{\Box\Gamma \succ \Box A}\,(\Box) \qquad \dfrac{\begin{matrix}D_1\\ (A)^n, \Sigma \succ B\end{matrix}}{(\Box A)^n, \Box\Sigma \succ \Box B}\,(\Box)}{\Box\Gamma, \Box\Sigma \succ \Box B}\,(\text{Mix})$$

where the subdeductions D_0 of $\Gamma \succ A$, D_1 of $(A)^n, \Sigma \succ B$ contains no mix, and $n > 0$. Then D is transformed into the following.

$$\dfrac{\dfrac{\begin{matrix}D_0 & D_1\\ \Gamma \succ A & (A)^n, \Sigma \succ B\end{matrix}}{\Gamma, \Sigma \succ B}\,(\text{Mix})}{\Box\Gamma, \Box\Sigma \succ \Box B}\,(\Box)$$

The case where the cut formula is $\sim\Box A$ is symmetric. ■

THEOREM 2. *The cut rule is eliminable in* $\mathbf{GK}^{+}_{*\mathbf{FDE}}$.

PROOF. This proof is substantially similar to the previous one, and, as before, we will show that (Mix) can be eliminated. We focus on one of the cases where the modal rules are used to obtain both premises.

Consider the following deduction D:

$$\dfrac{\dfrac{\begin{matrix}D_0\\ \Gamma \succ A, \sim\Delta\end{matrix}}{\Box\Gamma \succ \Box A, \sim\Box\Delta}\,(\Box^{+}) \qquad \dfrac{\begin{matrix}D_1\\ (A)^n, \Sigma \succ B, \sim\Theta\end{matrix}}{(\Box A)^n, \Box\Sigma \succ \Box B, \sim\Box\Theta}\,(\Box^{+})}{\Box\Gamma, \Box\Sigma \succ \Box B, \sim\Box\Delta, \sim\Box\Theta}\,(\text{Mix})$$

where the subdeductions D_0 of $\Gamma \succ A$, D_1 of $(A)^n, \Sigma \succ B$ contains no mix, and $n > 0$. Then D is transformed into the following.

$$\dfrac{\dfrac{\begin{matrix}D_0 & D_1\\ \Gamma \succ A, \sim\Delta & (A)^n, \Sigma \succ B, \sim\Theta\end{matrix}}{\Gamma, \Sigma \succ B, \sim\Delta, \sim\Theta}\,(\text{Mix})}{\Box\Gamma, \Box\Sigma \succ \Box B, \sim\Box\Delta, \sim\Box\Theta}\,(\Box^{+})$$

A symmetric argument can be used in the case where the cut formula is $\sim\Box A$. ■

Thus, two of the sequent systems defined in this section enjoy cut elimination. As we will see in the next section, all five systems are sound and complete with respect to appropriate classes of models.

5. Soundness and Completeness

In this section, we demonstrate the soundness and completeness of our sequent systems with respect to appropriate classes of models. Given that we will be interested in the adequacy of the proof systems defined in the previous section, for purposes of this section, let $\mathbf{L} \in \{\mathbf{K}_{*\mathbf{FDE}}, \mathbf{K}^{+}_{*\mathbf{FDE}}, \mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}, \mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}, \mathbf{K}\upsilon_{*\mathbf{FDE}}\}$ unless differently specified. We will now prove the soundness and completeness for each logic, but before that, we will introduce several definitions needed for proving completeness using the canonical model method.

5.1. Definitions for the Canonical Model Method

For sets of formulas Γ, Δ, we write $\Gamma \vdash_{\mathbf{L}} \Delta$ iff there are finite sets $\Sigma \subseteq \Gamma$ and $\Theta \subseteq \Delta$ such that $\mathbf{L} \vdash \Sigma \succ \Theta$.

Definition 31. A set Γ of formulas is an $\mathbf{L}$-theory iff if $\Gamma \vdash_{\mathbf{L}} A$, then $A \in \Gamma$.

An $\mathbf{L}$-theory is prime iff whenever $A \vee B \in \Gamma$, then either $A \in \Gamma$ or $B \in \Gamma$.

Definition 32. A pair (Γ, Δ) is $\mathbf{L}$-consistent iff there are no finite subsets $\Sigma \subseteq \Gamma$ and $\Theta \subseteq \Delta$ such that $\Sigma \vdash_{\mathbf{L}} \Theta$.

A pair (Γ, Δ) is $\mathbf{L}$-inconsistent iff it is not $\mathbf{L}$-consistent.

Given any $\mathbf{L}$-consistent pair (Γ, Δ), we can construct a prime extension Σ of Γ that is disjoint with Δ. We do this using the Belnap-Gabbay-Takeuti Prime Extension Lemma.

Lemma 33. *Given any* $\mathbf{L}$*-consistent pair* (Γ, Δ)*, there are* $\Sigma \supseteq \Gamma$ *and* $\Theta \supseteq \Delta$ *such that* Σ *is a prime theory,* $\Sigma \cup \Theta = \mathsf{Form}$*, and* (Σ, Θ) *is* $\mathbf{L}$*-consistent.*

Proof. See the proof of Restall [30, 94]. ∎

We will prove some general lemmas that will apply to all of our canonical models. First, let $W_{\mathbf{L}}$ be the set of prime $\mathbf{L}$-theories Γ such that $(\Gamma, \mathsf{Form} \setminus \Gamma)$ is $\mathbf{L}$-consistent. In most of the cases we are interested in, this can be simplified to the set of prime $\mathbf{L}$-theories, as we prove in the next lemma.

Lemma 34. *For* $\mathbf{L} \in \{\mathbf{K}_{*\mathbf{FDE}}, \mathbf{K}^{+}_{*\mathbf{FDE}}, \mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}, \mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}\}$*,* $W_{\mathbf{L}}$ *is the set of prime* $\mathbf{L}$*-theories.*

PROOF. $W_{\mathbf{L}}$ is clearly a subset of the set of prime $\mathbf{L}$-theories. Suppose Γ is a prime $\mathbf{L}$-theory such that $(\Gamma, \mathsf{Form} \setminus \Gamma)$ is not $\mathbf{L}$-consistent. We will set $\Delta = \mathsf{Form} \setminus \Gamma$. Since Γ is an $\mathbf{L}$-theory, if there is $A \in \Delta$ such that $\Gamma \vdash_{\mathbf{L}} A$, then $A \in \Gamma$, so $A \notin \Delta$, contradicting our assumption. Therefore, either there is a finite $\Sigma \subseteq \Gamma$ and $\Sigma \succ \emptyset$ is derivable or there is a finite $\Theta \subseteq \Delta$ and $\emptyset \succ \Theta$ is derivable. But, it is immediately clear by an inspection of the rules that neither $\emptyset \succ \Sigma$ nor $\Sigma \succ \emptyset$ is derivable in $\mathbf{GL}$, for any finite $\Sigma \subseteq \mathsf{Form}$. Therefore, there is no such theory Γ. ∎

$W_{\mathbf{K}\upsilon_{*\mathbf{FDE}}}$ is not the set of prime $\mathbf{K}\upsilon_{*\mathbf{FDE}}$-theories, however, as some prime theories, such as $\emptyset$ and Form, are not included. In what follows, we simply write W to mean $W_{\mathbf{L}}$ if the logic under consideration is clear from the context.

For all of our canonical models, the $*$ operation is defined in the same way.

- For $\Gamma \in W_{\mathbf{L}}$, $\Gamma^* = \{B \in \mathsf{Form} : {\sim}B \notin \Gamma\}$.

LEMMA 35. *For any* $\Gamma \in W_{\mathbf{L}}$, $\Gamma^* \in W_{\mathbf{L}}$.

PROOF. First, we show that Γ^* is a theory, i.e., $\Gamma^* \vdash B$ implies $B \in \Gamma^*$. Suppose that $\Gamma^* \vdash B$, i.e., there are finitely many formulas $A_1, \ldots, A_n$ such that $\vdash A_1, \ldots, A_n \succ B$ and $\{{\sim}A_1, \ldots, {\sim}A_n\} \cap \Gamma = \varnothing$. We show that $B \in \Gamma^*$, i.e., ${\sim}B \notin \Gamma$. Suppose for contradiction that ${\sim}B \in \Gamma$. By proposition 30, it follows from $\vdash A_1, \ldots, A_n \succ B$ that $\vdash {\sim}B \succ {\sim}A_1, \ldots, {\sim}A_n$ hence $\vdash {\sim}B \succ {\sim}A_1 \vee \cdots \vee {\sim}A_n$. Since Γ is a prime theory, we can find $1 \leqslant i \leqslant n$ such that ${\sim}A_i \in \Gamma$, which contradicts $\{{\sim}A_1, \ldots, {\sim}A_n\} \cap \Gamma = \varnothing$.

Second, we also need to show that if $A \vee B \in \Gamma^*$, then either $A \in \Gamma^*$ or $B \in \Gamma^*$. Suppose $A \notin \Gamma^*$ and $B \notin \Gamma^*$. Then ${\sim}A \in \Gamma$ and ${\sim}B \in \Gamma$. This implies ${\sim}(A \vee B) \in \Gamma$, so $A \vee B \notin \Gamma^*$.

When $\mathbf{L} = \mathbf{K}\nu_{*\mathbf{FDE}}$, we also need to show that $(\Gamma^*, \mathsf{Form} \setminus \Gamma^*)$ is consistent. To prove the consistency, assume for contradiction that there are finitely many formulas $A_1, \ldots, A_n \in \Gamma^*$ and $C_1, \ldots, C_m \notin \Gamma^*$ such that $\vdash A_1, \ldots, A_n \succ C_1, \ldots, C_m$. By $A_i \in \Gamma^*$ and $C_j \notin \Gamma^*$, we get ${\sim}A_i \notin \Gamma$ and ${\sim}C_j \in \Gamma$, respectively. By Proposition 30, $\vdash {\sim}C_1, \ldots, {\sim}C_m \succ {\sim}A_1, \ldots, {\sim}A_n$ hence $\vdash {\sim}C_1 \wedge \cdots \wedge {\sim}C_m \succ {\sim}A_1 \vee \cdots \vee {\sim}A_n$. Since Γ is a theory and ${\sim}C_i \in \Gamma$, we get ${\sim}A_1 \vee \cdots \vee {\sim}A_n \in \Gamma$ hence ${\sim}A_i \in \Gamma$ for some i. This is a contradiction with ${\sim}A_i \notin \Gamma$. ∎

LEMMA 36. *For any prime theory* $\Gamma \in W_{\mathbf{L}}$, $\Gamma^{**} = \Gamma$ *holds.*

PROOF. We proceed as follows: $B \in \Gamma$ iff ${\sim}{\sim}B \in \Gamma$ iff ${\sim}B \notin \Gamma^*$ iff $B \in \Gamma^{**}$, where the first equivalence holds since Γ is a theory. ∎

5.2 $\mathbf{GK}_{*\mathbf{FDE}}$ and $\mathbf{GK}^{+}_{*\mathbf{FDE}}$

We now begin the soundness and completeness proofs with the weaker logics, $\mathbf{GK}_{*\mathbf{FDE}}$ and $\mathbf{GK}^{+}_{*\mathbf{FDE}}$, starting with the former. First, we define validity for multi-succedent sequents.

Definition 37. A sequent $\Gamma \succ \Delta$ is valid in the class of $\mathbf{L}$-models, $\Gamma \models_{\mathbf{L}} \Delta$, iff for all $\mathbf{L}$-models $\langle W, R, *, V\rangle$, for all $w \in W$, if $I(w, A) = 1$, for all $A \in \Gamma$, then $I(w, B) = 1$, for some $B \in \Delta$.

Let us prove soundness of $\mathbf{GK}_{*\mathbf{FDE}}$ with respect to $\mathbf{K}_{*\mathbf{FDE}}$-models.

Theorem 3. *If* $\Gamma \vdash_{\mathbf{K}_{*\mathbf{FDE}}} \Delta$, *then* $\Gamma \models_{\mathbf{K}_{*\mathbf{FDE}}} \Delta$.

Proof. The proof is by induction on the structure of proofs. The non-modal cases are routine. The modal cases are covered by proposition 27. ■

For completeness, we need to define the canonical $\mathbf{K}_{*\mathbf{FDE}}$-model. To do this, we will use the notation $\Gamma^{-} = \{B \in \mathsf{Form} : \Box B \in \Gamma\}$. Since we are only concerned with one logic at a time for each canonical model construction, we will leave the subscripts off the parts of the canonical models, letting context settle what logic is under discussion.

- W is the set of prime $\mathbf{K}_{*\mathbf{FDE}}$-theories.
- $\Gamma R \Delta$ iff $\Gamma^{-} \subseteq \Delta$.
- $\Gamma^{*} = \{B \in \mathsf{Form} : {\sim}B \notin \Gamma\}$.
- $V(p) = \{\Gamma \in W : p \in \Gamma\}$.

The main thing to verify here is that the truth lemma works. For that, we need an existence lemma.

Lemma 38. *For all formulas* A, *if* $\Box A \notin \Gamma$, *then there is some* Δ *such that* $A \notin \Delta$ *and* $\Gamma R \Delta$.

Proof. The proof is routine, following that of Restall [30, 255-256]. ■

From this, we can prove the truth lemma.

Lemma 39. *For all formulas* $A \in \mathsf{Form}$ *and* $\Gamma \in W$, $I(\Gamma, A) = 1$ *iff* $A \in \Gamma$.

Proof. The proof is by a standard induction on the construction of A. ■

The completeness theorem then follows. Thus, we have the following theorem.

Theorem 4. *If* $\Gamma \models_{\mathbf{K}_{*\mathbf{FDE}}} A$, *then* $\Gamma \vdash_{\mathbf{K}_{*\mathbf{FDE}}} A$.

PROOF. Assume $\Gamma \nvdash_{\mathbf{K}_{*\mathbf{FDE}}} A$, so $(\Gamma, \{A\})$ is a $\mathbf{K}_{*\mathbf{FDE}}$-consistent pair. Extend Γ to a prime theory $\Sigma \in W$ such that $A \notin \Sigma$. By the preceding lemma, $I(\Sigma, A) = 0$, so $\Gamma \not\models_{\mathbf{K}_{*\mathbf{FDE}}} A$, as desired. ■

Next, we turn to $\mathbf{GK}^{+}_{*\mathbf{FDE}}$ and prove soundness for it.

THEOREM 5. *If* $\Gamma \vdash_{\mathbf{K}^{+}_{*\mathbf{FDE}}} \Delta$, *then* $\Gamma \models_{\mathbf{K}^{+}_{*\mathbf{FDE}}} \Delta$.

PROOF. The modal cases are covered by proposition 27 together with proposition 6. ■

Now, we demonstrate completeness for $\mathbf{GK}^{+}_{*\mathbf{FDE}}$ with respect to $\mathbf{K}^{+}_{*\mathbf{FDE}}$-models. To prove this, we need to change the canonical model in the following ways.

- W is the set of prime $\mathbf{K}^{+}_{*\mathbf{FDE}}$-theories.
- $\Gamma R \Delta$ iff $\Gamma^{-} \subseteq \Delta$ and $(\Gamma^{*})^{-} \subseteq \Delta^{*}$.

Then the (R^{*}) condition is satisfied by definition. We also note that

For completeness, we show the following lemma analogous to the existence lemma above.

LEMMA 40. *In the canonical* $\mathbf{K}^{+}_{*\mathbf{FDE}}$*-model, for all* $\Gamma \in W$, *if* $\Box A \notin \Gamma$, *then there is a* $\Delta \in W$ *such that* $\Gamma R \Delta$ *and* $A \notin \Delta$.

PROOF. Suppose that $\Box A \notin \Gamma$. We show that $\Gamma^{-} \nvdash_{\mathbf{K}^{+}_{*\mathbf{FDE}}} A, \{\sim C \mid \sim\Box C \notin \Gamma\}$. Assume for contradiction that $\Gamma^{-} \vdash_{\mathbf{K}^{+}_{*\mathbf{FDE}}} A, \{\sim C \mid \sim\Box C \notin \Gamma\}$. This means that

$$\mathbf{K}^{+}_{*\mathbf{FDE}} \vdash B_1, \ldots, B_n \succ A, \sim C_1, \ldots, \sim C_m$$

for some $B_1, \ldots, B_n \in \Gamma^{-}$ and $\sim C_1, \ldots, \sim C_m \in \{\sim C \mid \sim\Box C \notin \Gamma\}$. By applying $(\Box^{+})$, we obtain

$$\mathbf{K}^{+}_{*\mathbf{FDE}} \vdash \Box B_1, \ldots, \Box B_n \succ \Box A, \sim\Box C_1, \ldots, \sim\Box C_m.$$

Then since Γ is a prime theory, we can conclude from $\Box B_i \in \Gamma$ $(1 \leqslant i \leqslant n)$ and $\sim\Box C_i \in \Gamma$ $(1 \leqslant j \leqslant m)$ that $\Box A \in \Gamma$. A contradiction. Thus, $\Gamma^{-} \nvdash_{\mathbf{K}^{+}_{*\mathbf{FDE}}} A, \{\sim C \mid \sim\Box C \in \Gamma\}$.

Since Lemma 33 holds good also for $\mathbf{GK}^{+}_{*\mathbf{FDE}}$, we have a prime theory Δ such that $\Gamma^{-} \subseteq \Delta$ and $\{A\} \cup \{\sim C \mid \sim\Box C \in \Gamma\} \subseteq \mathsf{Form} \setminus \Delta$. By $\{\sim C \mid \sim\Box C \in \Gamma\} \subseteq \mathsf{Form} \setminus \Delta$, it is immediate that $(\Gamma^{*})^{-} \subseteq \Delta^{*}$. ■

The rest of the completeness proof can be done the same as above. Thus we have the following theorem.

THEOREM 6. *If* $\Gamma \models_{\mathbf{K}^{+}_{*\mathbf{FDE}}} A$, *then* $\Gamma \vdash_{\mathbf{K}^{+}_{*FDE}} A$.

5.3 $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$ and $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$

We turn now to the **S5**-ish logics, starting with $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$. Before we prove soundness, we will prove a lemma.

LEMMA 41. $\Box A \vee \Box B \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box(\Box A \vee \Box B)$.

PROOF. Since $\Box A \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box A \vee \Box B$, by proposition 27, $\Box\Box A \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box(\Box A \vee \Box B)$. As $\Box A \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box\Box A$, we have $\Box A \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box(\Box A \vee \Box B)$. Similar reasoning establishes $\Box B \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box(\Box A \vee \Box B)$, whence $\Box A \vee \Box B \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box(\Box A \vee \Box B)$. ■

THEOREM 7. *If* $\Gamma \vdash_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Delta$, *then* $\Gamma \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Delta$.

PROOF. We will do two of the modal cases, as the other two non-modal cases are handled by straightforward adaptations of the arguments.

Suppose the final rule is $(\sim\Box R)$.

$$\frac{\Gamma \succ \Delta, \sim A}{\Gamma \succ \Delta, \sim\Box A}\ (\sim\Box R)$$

We assume $\Gamma \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Delta, \sim A$. Suppose $\langle W, R, *, V\rangle$ is a $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$-model with $w \in W$ such that $I(w, B) = 1$, for $B \in \Gamma$, $I(w, C) = 0$, for $C \in \Delta$, and $I(w, \sim\Box A) = 0$. Therefore, $I(w^*, \Box A) = 1$, so for all $x \in W$, if $w^* R x$, then $I(x, A) = 1$. Since R is reflexive, $w^* R w^*$, so $I(w^*, A) = 1$. But this implies, $I(w, \sim A) = 0$, which together with the other assumptions means $\Gamma \not\models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Delta, \sim A$, which is a contradiction.

Next, suppose the final rule is $(\Box R)$.

$$\frac{\Box\Gamma \succ \Box\Delta, A}{\Box\Gamma \succ \Box\Delta, \Box A}\ (\Box R)$$

Assume that $\Box\Gamma \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box\Delta, A$. This implies $\Box\Gamma \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \bigvee \Box\Delta, A$, and by lemma 41, $\Box\Gamma \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box\bigvee \Box\Delta, A$. This in turn implies $\Box\Gamma \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} A \vee \Box\bigvee \Box\Delta$, so by proposition 27, $\Box\Box\Gamma \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box(A \vee \Box\bigvee \Box\Delta)$. As $\Box(A \vee \Box B) \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box A \vee \Box B$, for all formulas A and B, we have that $\Box\Box\Gamma \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box A \vee \Box\bigvee \Box\Delta$. The desired conclusion that $\Box\Gamma \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box A, \Box\Delta$ then follows straightforwardly.

The remaining two modal cases are similar to the preceding. ■

To define the canonical model for $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$, we make some adjustments to the previously defined canonical models.

- W is the set of prime $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$-theories.
- $\Gamma R \Delta$ iff $\Gamma^- \subseteq \Delta$ and $\Delta^- \subseteq \Gamma$.

The other parts of the model are defined as before. It remains to verify that the structure is a $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$-model and that it can provide an appropriate countermodel.

We start by showing that R obeys the required conditions.

LEMMA 42. *R is symmetric, reflexive, and transitive.*

PROOF. Symmetry is immediate from the definition of R.

Reflexivity follows from $\Box A \vdash_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} A$.

Transitivity follows from $\Box A \vdash_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Box\Box A$. ■

We follow the strategy of Standefer and Mares [44] for proving the existence lemma.

LEMMA 43. *For all $\Gamma \in W$, if $\Box A \notin \Gamma$, then there is a $\Delta \in W$ such that $\Gamma R \Delta$ and $A \notin \Delta$.*

PROOF. Suppose $\Box A \notin \Gamma$. We construct an appropriate Δ. Let $\Sigma = \{\Box D \in \mathsf{Form} : D \notin \Gamma\}$, and let Θ be $\Sigma \cup \{A\}$.

Suppose $\Gamma^- \vdash_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Theta$. This means that

$$\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}} \vdash B_1, \ldots, B_n \succ A, \Box D_1 \ldots, \Box D_m,$$

for some $B_i \in \Gamma^-$ and some $D_j \in \Sigma$. From this we use $\Box L$ followed by $\Box R$ to obtain

$$\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}} \vdash \Box B_1, \ldots, \Box B_n \succ \Box A, \Box D_1, \ldots, \Box D_m.$$

From this, we obtain

$$\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}} \vdash \Box B_1, \ldots, \Box B_n \succ \Box A \vee (\Box D_1 \vee \ldots \vee \Box D_m).$$

Since $\Box B_i \in \Gamma$, for $i \leq n$, it follows that $\Box A \vee (\Box D_1 \vee \ldots \vee \Box D_m) \in \Gamma$. But, Γ is prime and for $j \leq m$, $\Box D_j \notin \Gamma$. So, $\Box A \in \Gamma$, but this contradicts the initial assumption. Therefore, $\Gamma^- \nvdash_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} \Theta$.

This means that (Γ^-, Θ) is $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$-consistent, so using lemma 33, we can obtain $\Xi \supseteq \Gamma^-$ such that (Ξ, Θ) is $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$-consistent. We need to show that $\Gamma R \Xi$.

To show $\Gamma R \Xi$, we need to show that $\Xi^- \subseteq \Gamma$, as $\Gamma^- \subseteq \Xi$ by construction. So, assume $\Box C \in \Xi$ but $C \notin \Gamma$. Then, $\Box C \in \Theta$, but this contradicts the fact that (Ξ, Θ) is $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$-consistent. Therefore, $\Xi^- \subseteq \Gamma$.

Since (Ξ, Θ) is $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$-consistent, $A \notin \Xi$. Thus, $\Gamma R \Xi$ and $A \notin \Xi$, as desired. ■

LEMMA 44. *For all $A \in \mathsf{Form}$, $A \in \Gamma$ iff $I(\Gamma, A) = 1$.*

PROOF. The cases where A is p, $B \wedge C$, $B \vee C$, or $\sim B$ are standard. One direction of the case where A is $\Box B$ is handled by the preceding lemma, and the other direction is standard. ■

Putting these pieces together, we obtain completeness.

THEOREM 8. *If* $\Gamma \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} A$, *then* $\Gamma \vdash_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}} A$.

We now demonstrate soundness and completeness for $\mathbf{GK}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$ with respect to $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$-models.

THEOREM 9. *If* $\Gamma \vdash_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}^+}} \Delta$, *then* $\Gamma \models_{\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}^+}} \Delta$.

PROOF. We only show that $(\Box^{+}R)$ preserves validity. Suppose $\sim\Box\Pi, \Box\Gamma \models_{\mathbf{K}^{+}\rho\sigma\tau_{*\mathbf{FDE}}} \Box\Delta, A, \sim\Box\Sigma$. Fix any $\mathbf{K}^{+}\rho\sigma\tau_{*\mathbf{FDE}}$-model $M = \langle W, R, *, V\rangle$ and $w \in W$. Suppose all the following:

- $I(w, \sim\Box B) = 1$ for all $B \in \Pi$, $I(w, \Box C) = 1$ for all $C \in \Gamma$,
- $I(w, \Box D) \neq 1$ for all $D \in \Delta$, and $I(w, \sim\Box E) \neq 1$ for all $E \in \Sigma$.

To show that $I(w, \Box A) = 1$, let us fix any v such that wRv. Our goal is to show that $I(v, A) = 1$. By wRv, we obtain the following: $I(v, \Box C) = 1$ for all $C \in \Gamma$, and $I(v, \Box D) \neq 1$ for all $D \in \Delta$, by transitivity and Euclideanness of R, respectively. We also get the following: $I(w^*, \Box B) \neq 1$ for all $B \in \Pi$, and $I(w^*, \Box E) = 1$ for all $E \in \Sigma$. It follows from wRv that w^*Rv^*. Thus we can also establish the following: $I(v^*, \Box B) \neq 1$ for all $B \in \Pi$, and $I(v^*, \Box E) = 1$ for all $E \in \Sigma$ by definition. These imply the following: $I(v, \sim\Box B) = 1$ for all $B \in \Pi$, and $I(v, \sim\Box E) \neq 1$ for all $E \in \Sigma$. Finally, we conclude from $\sim\Box\Pi, \Box\Gamma \models_{\mathbf{K}^{+}\rho\sigma\tau_{*\mathbf{FDE}}} \Box\Delta, A, \sim\Box\Sigma$ that $I(v, A) = 1$. ■

This proof system satisfies the conditions for lemma 33. We construct the canonical $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$-model, with the only change being that $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$-theories are used rather than $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$-theories.

We need to verify that in the new model, R satisfies the (R^*) condition. First of all, the following sequents are derivable in the system under consideration:

$$\dfrac{\dfrac{\dfrac{\sim A \succ \sim A}{\sim A \succ \sim\Box A}\,(\sim\Box R) \qquad \dfrac{\sim\Box A \succ \sim\Box A}{\sim\Box A \succ \Box\sim\Box A}\,(\Box^{+}R)}{\sim A \succ \Box\sim\Box A}\,(\text{Cut})}{}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Box A \succ \Box A}{\sim\sim\Box A \succ \Box A}\,(\sim\sim L)}{\sim\Box\sim\Box A \succ \Box A}\,(\sim\Box^{+}L) \qquad \dfrac{A \succ A}{\Box A \succ A}\,(\Box L)}{\sim\Box\sim\Box A \succ A}\,(\text{Cut})}{}$$

LEMMA 45. *In the canonical* $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$*-model, if* $\Gamma R\Delta$*, then* $\Gamma^* R\Delta^*$*, therefore,* $\Gamma R\Delta$ *iff* $\Gamma^* R\Delta^*$*.*

PROOF. The latter statement follows from Lemma 36. For the former, suppose that $\Gamma R\Delta$. For reductio, assume for some A, $\Box A \in \Gamma^*$ and $A \notin \Delta^*$. The latter implies $\sim A \in \Delta$. Since $\sim A \vdash_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} \Box\sim\Box A$, it follows that $\Box\sim\Box A \in \Delta$. This implies $\sim\Box A \in \Delta^-$. From the assumption that $\Gamma R\Delta$, it follows that $\sim\Box A \in \Gamma$. Therefore, $\Box A \notin \Gamma^*$, which is a contradiction. Thus, $\Gamma^- \subseteq \Delta$. The proof that $\Delta^- \subseteq \Gamma$ is similar. ■

The rest of the results concerning the canonical model carry over from before. Therefore, we can conclude with the following theorem.

THEOREM 10. *If* $\Gamma \models_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} A$*, then* $\Gamma \vdash_{\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}} A$*.*

5.4 $\mathbf{K}\upsilon_{*\mathbf{FDE}}$

Finally, we demonstrate soundness and completeness for $\mathbf{GK}\upsilon_{*\mathbf{FDE}}$ with respect to star models with a universal relation. We begin with soundness.

THEOREM 11. *If* $\Gamma \vdash_{\mathbf{K}\upsilon_{*\mathbf{FDE}}} \Delta$*, then* $\Gamma \models_{\mathbf{K}\upsilon_{*\mathbf{FDE}}} \Delta$*.*

PROOF. The result is straightforward from lemma 21. ■

For the completeness proof, we will first prove a few lemmas.

LEMMA 46. *The following are derivable in* $\mathbf{GK}\upsilon_{*\mathbf{FDE}}$*.*

- $\sim\Box A, \Box A\succ$
- $\succ\Box A, \sim\Box A$

PROOF. These can each be proved by application of the $(\sim\Box)$ rules to $\Box A\succ \Box A$. ■

LEMMA 47. *Let* Γ *be a* $\mathbf{K}\upsilon_{*\mathbf{FDE}}$*-theory, and suppose that* $\sim C \in \Gamma$*, for some formula* C *such that* $\mathbf{K}\upsilon_{*\mathbf{FDE}} \vdash \succ C$*. Then,* $\Gamma = \mathsf{Form}$*.*

PROOF. Since the contraposition rule is admissible in $\mathbf{GK}\upsilon_{*\mathbf{FDE}}$, if $\mathbf{K}\upsilon_{*\mathbf{FDE}} \vdash \succ C$ $\sim C \in \Gamma$, then $\mathbf{K}\upsilon_{*\mathbf{FDE}} \vdash \sim C\succ$. Using (WR), $\mathbf{K}\upsilon_{*\mathbf{FDE}} \vdash \sim C \succ B$, for all B. Since Γ is a $\mathbf{K}\upsilon_{*\mathbf{FDE}}$-theory, it follows that $\Gamma = \mathsf{Form}$. ■

Unlike in the previous proofs in this section, we will not define a single canonical model. For each choice of unprovable sequent, $\Gamma\succ\Delta$, we will obtain a potentially distinct canonical model. So, for the remainder of this section, assume that $\Gamma_0 \nvdash_{\mathbf{K}\upsilon_{*\mathbf{FDE}}} \Delta_0$, for some Γ_0 and Δ_0. Using lemma 33, we can obtain a prime $\mathbf{K}\upsilon_{*\mathbf{FDE}}$-theory Ω such that $(\Omega, \mathsf{Form} \setminus \Omega)$ is a $\mathbf{K}\upsilon_{*\mathbf{FDE}}$-consistent pair extending (Γ_0, Δ_0).

Let W be the set of prime $\mathbf{K}\upsilon_{*\mathbf{FDE}}$-theories Γ such that $(\Gamma, \mathsf{Form} \setminus \Gamma)$ is $\mathbf{K}\upsilon_{*\mathbf{FDE}}$-consistent. Define $\Sigma R \Theta$ as $\Sigma^- \subseteq \Theta$ and $\Theta^- \subseteq \Sigma$. We can then define the desired canonical model as follows.

- $W_\Omega = \{\Sigma \in W : \Omega R \Sigma\}$.
- $\Gamma R_\Omega \Delta$ iff $\Gamma R \Delta$.

The canonical $*$ and V are defined as before.

We need to show that R_Ω is universal on W_Ω and that $*$ is well-defined in W_Ω.

Lemma 48. *R_Ω is the universal relation on W_Ω.*

Proof. Suppose that $\Gamma, \Delta \in W_\Omega$. Then, $\Omega R \Gamma$ and $\Omega R \Delta$. Since $\mathbf{GK}\upsilon_{*\mathbf{FDE}}$ extends $\mathbf{GK}\rho\sigma\tau_{*\mathbf{FDE}}$, R is an equivalence relation, by our earlier arguments. Therefore, $\Gamma R_\Omega \Delta$, as desired. ∎

Next, we show that $*$ is well-defined.

Lemma 49. *For $\Gamma \in W$ and $A \in \mathsf{Form}$, $\Box A \in \Gamma$ iff $\sim\Box A \notin \Gamma$.*

Proof. By lemma 46, $\mathbf{K}\upsilon_{*\mathbf{FDE}} \vdash \succ \Box A \vee \sim\Box A$. So, for all $\Gamma \in W$, $\Box A \vee \sim\Box A \in \Gamma$. Similarly, $\mathbf{K}\upsilon_{*\mathbf{FDE}} \vdash \Box A \wedge \sim\Box A \succ$.

Suppose $\sim\Box A \notin \Gamma$. Since Γ is prime, $\Box A \in \Gamma$ or $\sim\Box A \in \Gamma$. It follows that $\Box A \in \Gamma$.

Suppose that $\Box A \in \Gamma$. For reductio, suppose that $\sim\Box A \in \Gamma$. It then follows, since Γ is a theory, that $\sim\Box A \wedge \Box A \in \Gamma$. By lemma 47, this implies that $\Gamma = \mathsf{Form}$, but then $\Gamma \notin W$, contradicting our initial assumption. Therefore, $\sim\Box A \notin \Gamma$. ∎

Lemma 50. *For $\Gamma \in W_\Omega$, $\Gamma^* \in W_\Omega$.*

Proof. Suppose $\Omega R \Gamma$. We show that $\Gamma R \Gamma^*$, which is sufficient for $\Gamma^* \in W_\Omega$, as R is an equivalence relation.

Suppose that $A \in \Gamma^-$, so $\Box A \in \Gamma$. It follows that $\sim\Box A \notin \Gamma^*$. By lemma 49, $\Box A \in \Gamma^*$, so $A \in \Gamma^*$, as $\mathbf{K}\upsilon_{*\mathbf{FDE}} \vdash \Box A \succ A$.

Suppose that $A \in \Gamma^{*-}$. Then $\Box A \in \Gamma^*$. So, $\sim\Box A \notin \Gamma$. By lemma 49, $\Box A \in \Gamma$. Just as in the previous case, it follows that $A \in \Gamma$, as desired.

Therefore, $\Gamma R \Gamma^*$. Since R is an equivalence relation, $\Omega R \Gamma^*$. Therefore, $\Gamma^* \in W_\Omega$. ∎

Next, we have a form of the existence lemma.

LEMMA 51. *For all* $\Gamma \in W_\Omega$ *and* $A \in \mathsf{Form}$*, if* $\Box A \notin \Gamma$*, there there is a* $\Delta \in W_\Omega$ *such that* $\Gamma R_\Omega \Delta$ *and* $A \notin \Delta$*.*

PROOF. It follows from lemma 43 that there is a $\Delta \in W$ such that $\Gamma R \Delta$. Since R is an equivalence relation, $\Omega R \Delta$, which suffices for $\Gamma R_\Omega \Delta$. ■

LEMMA 52. *For all* $\Gamma \in W_\Omega$ *and* $A \in \mathsf{Form}$*,* $A \in \Gamma$ *iff* $I(\Gamma, A) = 1$*.*

PROOF. The proof proceeds by an induction similar to the previous two cases. The previous lemma takes care of one direction of the $\Box$ case, and a standard argument takes care of the other direction. ■

With these pieces in hand, we can claim completeness.

THEOREM 12. *If* $\Gamma \models_{\mathbf{K}\upsilon_{*}\mathbf{FDE}} \Delta$*, then* $\Gamma \vdash_{\mathbf{K}\upsilon_{*}\mathbf{FDE}} \Delta$*.*

Thus, for all five of the sequent systems presented in the previous section, we have proven soundness and completeness with respect to an appropriate class of models.

6. Concluding Remarks

In this paper, we have defined modal logics over the Australian plan models for **FDE** and the American plan models for **FDE**. We have shown that when the Australian plan models obey an extra condition, they are equivalent to the American plan models. When we considered **S5**-type logics in particular, we found that there were three distinct logics that emerged. We provided cut-free sequent systems for two of the modal logics, and showed five logics to be sound and complete with respect to appropriate classes of models. We think that further exploration of modal logics over **FDE** is promising, and such exploration will highlight further differences between the Australian and American plans for **FDE**.

In the basic, non-modal vocabulary, the two plans for negation in **FDE** are equivalent. Models for one can be converted into models for the other. Once one allows modal operators into the language, differences emerge. In particular, the Australian plan results in a weaker logic. The difference in strength can be remedied by restricting to a subclass of models, namely those that obey the (R^*) condition. If one does this, then models for one plan can be transformed into models for the other.

There are limits to the coincidence between the two plans. If one considers the class of models where R is the universal relation, then the two plans come apart again. This time, however, the Australian plan results in the stronger logic. In a model with a universal relation, $\Box A$ is forced to be true everywhere or nowhere, which means that $\Box A$ and $\sim\Box A$ cannot both be true at any world. By contrast, the American plan allows that $\Box A$ and $\sim\Box A$ can both be true (and false) at a world. It seems that adapting the American plan models to match the logic of the $\mathbf{K}\upsilon_{*\mathbf{FDE}}$-models would require adopting different truth and falsity conditions for $\Box$, but we will leave further investigation of this idea for future work.

We have, in this paper, considered only logics with one additional modal operator, $\Box$. We expect that the relationships between the two plans for negation to be further complicated if additional modal operators are added.

One application of our results that we think is promising is epistemic logic. In epistemic logic, $\Box$ is interpreted as a knowledge operator, so that $\Box A$ says that the agent knows A.[17] In epistemic logic, the standard starting point is to use Kripke models where R is an equivalence relation, so that each equivalence class represents sets of worlds that the agent can distinguish.[18] The $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$-models and $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$-models use equivalence relations, and they give two distinct epistemic logics. While $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$ is similar to **S5**-based epistemic logic, $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$ lacks analogs of some of the principles that have been criticized as too strong for logics of knowledge, namely $\sim B \models \Box\sim\Box B$ and $\sim\Box B \models \Box\sim\Box B$.[19] These are analogs of the **(B)** and **(5)** axioms. While both logics fall well short of standard **S5** epistemic logic, they seem promising as starting points for investigating epistemic logic in the context of **FDE**.[20]

The invalidity of the **(B)** and **(5)** axioms in $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$ is one reason it is a promising candidate for applications of epistemic logic. Beyond that, both $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$ and $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$ have potentially interesting features as epistemic logics related to issues of logical omniscience.[21] There are a cluster of issues

[17] See [17,29,46] for introductions to epistemic logic in the setting of classical logic. See [15, ch. 5] for general discussion of epistemic logic. We restrict discussion here to the single agent case to simplify the presentation.

[18] See [37], among others, for critical discussion of the use of equivalence relations to model knowledge. For more on equivalence relations in relevant logics, see [41,44].

[19] See [12,15,47] for objections to these principles.

[20] For an alternative approach to using **FDE** in epistemic logic, see [45].

[21] For discussion of logical omniscience, see [10,11,15,49].

under the banner of logical omniscience, all of which deal with agents, intuitively, knowing too much. We will highlight some of the features of the logics that bear on these issues.

One form of omniscience is that agents are forced to know logical truths. Since neither logic has logical truths, there are no formulas that the logics would force every agent to know. The logics are closed under the rule of monotonicity, if $A \models B$, then $\Box A \models \Box B$, so agents' knowledge would be closed under logical entailment. Because of the variable-sharing property enjoyed by the logics, if an agent only knows claims involving p, closure of the agent's knowledge under the logic will not result in them knowing something involving only q. The reason is that if the only atom in A is p and the only atom in B is q, the argument $A \succ B$ will be invalid in both logics, on pain of violating the variable-sharing property. Further, the failure of the analog of **(K)**, $\Box(\sim A \vee B), \Box A \succ \Box B$, in both logics means that knowledge need not be closed under known material implications, which contrasts with standard, classically-based epistemic logics. Thus, while some issues of logical omniscience arise for both $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$ and $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$, both avoid some issues of omniscience.

We will close by noting two questions that we have left open.

(Q1) Is there a cut-free sequent system for $\mathbf{K}\rho\sigma\tau_{*\mathbf{FDE}}$, $\mathbf{K}\rho\sigma\tau^{+}_{*\mathbf{FDE}}$, or $\mathbf{K}\upsilon_{*\mathbf{FDE}}$?

The sequent systems we defined for the three logics above all fail to enjoy cut elimination. Given the known limitations of the basic sequent systems for modal logics, it seems plausible that non-standard sequent systems, such as hypersequent systems, could provide cut-free proof systems for the logics. A related question, which we also leave open is whether the logics obey analytic cut elimination.

(Q2) Do any of the logics enjoy any interpolation properties?

We have not attempted to investigate interpolation properties. We hope to do so in future work.

Acknowledgements. The ideas for this paper were conceived at the Taipei-Kyoto workshop on the Logic of Agent, held at National Taiwan University, and funded by an SP+ Grant from Kyoto University. Shawn Standefer's and Ren-June Wang's research was supported by the National Science

and Technology Council of Taiwan grant 114-2923-H-194-001-MY3. Hitoshi Omori thanks the coauthors for picking up the ideas that he presented at the above workshop, and also the audience for comments at the *Bochum Nonclassical Logic Workshop III* where he presented the semantic part of the results. Finally, but not the least, we would like to thank the referees for their careful reading and helpful comments.

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FDE-based Modal Logics

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