Ignorance and the possibility of error in relevant epistemic logic

Abstract

In this paper, I will present two approaches to epistemic logic in the setting of relevant logics. One uses the framework of equivalence classes representing indistinguishability, found in much work in epistemic logics. The other, does not use equivalence classes, but is more common in the area of relevant logics. I will argue that the former has many advantages over the latter while avoiding some of the standard criticisms leveled against the use of equivalence relations in classically based epistemic logic.

1 Introduction

One conception of knowledge in epistemic logics is that something is known if it is true in all epistemic alternatives. The epistemic alternatives at a given world, in turn, are the worlds that the agent cannot distinguish, given their background information. This idea of indistinguishability is formalized using equivalence relations in Kripke models. This modeling is in intuitive and powerful, and it is arguably the default modeling epistemic logic.[1] Nonetheless, the modeling and its logic have faced criticisms. In this paper, I will examine this conception of knowledge against the backdrop of relevant logics, in particular the logic R. I will argue that in the context of R, this conception of knowledge can evade many of the criticisms leveled against it. I further identify three related, plausible epistemic logics and comment on their relations. To begin, in §2, I provide background on a standard approach to epistemic logics using equivalence classes. In §3, I present the frames and axioms for R. In §4, I present three plausible epistemic extensions of R, focusing on two defined in terms of equivalence classes.

[1] See Meyer and van der Hoek (1995), van Ditmarsch et al. (2015), or Rendsvig et al. (2023).

Finally, in §5, I summarize the results of the earlier sections and highlight ways in which the logics relate to issues of logical omniscience.

2 Epistemic logics

Epistemic logics are most often studied with classical logic as the base logic.[2] The main results of this paper will concern an alternative base logic, but it will be useful to have the backdrop of standard epistemic logics in place before proceeding. We will throughout work in a propositional language $\mathcal{L}$ with the connectives $\{\neg, \to, \vee, \wedge, \Box\}$ and a countably infinite set of atoms $\mathsf{At} = \{p, q, r, \ldots\}$. The singulary operator $\Box$ will be the knowledge operator of the epistemic logics. We will define the possibility operator, $\Diamond$, as $\neg\Box\neg$.

The equivalence conception of knowledge is formalized using equivalence relations in Kripke frames.[3] Kripke frames with equivalence relations are the only sorts of Kripke frames used in this paper, so I will call them *classical partition frames*.

Definition 2.1 (Equivalence relation)**.** *A binary relation* S *on a set* X *is an equivalence relation iff it satisfies satisfies the three conditions*

- $\forall x \in X$, Sxx *(reflexivity),*
- $\forall x, y \in X$, *if* Sxy, *then* Syx *(symmetry), and*
- $\forall x, y, z \in X$, *if* Sxy *and* Syz, *then* Sxz *(transitivity).*

For an equivalence relation S *on a set* X*, we will define the notation* $[w] = \{x \in X : Swx\}$*. The sets* $[w]$ *are equivalence classes.*

Definition 2.2 (Classical partition frames)**.** *A classical partition frame is a pair* $\langle W, S\rangle$ *where* $W \neq \emptyset$ *and* S *is an equivalence relation on* W*.*

The relation S represents indiscernibility from the agent's point of view. Given the agent's background information, the agent cannot distinguish any of the worlds in a given equivalence class. These worlds are all the epistemic alternatives from any world in that class.

To define models, we add a valuation function.

[2] See van Ditmarsch et al. (2007) and Humberstone (2016, ch. 5) for some examples and references.

[3] In Kripke frames, it is more common to use 'R' for the binary relation, but we will reserve this for the ternary relation of the frames for relevant logics.

Definition 2.3 (Partition models)**.** *Let* $\langle W, S\rangle$ *be a classical partition frame.* $\langle W, S, V\rangle$ *is a partition model where* $V : \mathsf{At} \mapsto \wp W$.

The verification relation, $\Vdash$, for the language is defined inductively as follows.

- $w \Vdash p$ iff $w \in V(p)$
- $w \Vdash \neg B$ iff $w \nVdash B$
- $w \Vdash B \wedge C$ iff $w \Vdash B$ and $w \Vdash C$
- $w \Vdash B \vee C$ iff $w \Vdash B$ or $w \Vdash C$
- $w \Vdash B \rightarrow C$ iff $w \nVdash B$ or $C \Vdash$
- $w \Vdash \Box B$ iff for all $x \in [w]$, then $x \Vdash B$

Given the definition of models, we can define validity.

Definition 2.4 (Validity)**.** *A formula* B *is valid on the class classical partition frames iff for all models built on a classical partition frame, for all* $w \in W$, $w \Vdash B$.

The logic that results from this definition is S5. An axiomatization of S5 can be had by adding the following axioms and rule to classical logic, including the rule *modus ponens*. I will use '$A_1, \ldots, A_n \Rightarrow B$' for the rule from the formulas $A_1, \ldots, A_n$ to the conclusion B.[4]

- $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$ (K)
- $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$ (Agg)
- $\Box A \rightarrow A$ (T)
- $\Box A \rightarrow \Box\Box A$ (4)
- $\neg\Box A \rightarrow \Box\neg\Box A$ (5)
- $A \rightarrow \Box\neg\Box\neg A$ (B)

[4] The two rules in this list are, in Smiley's terminology, rules of proof, taking one from theorems to theorems. Rules of proof are contrasted with rules of inference, which take one from truths to truths. See Humberstone (2010) for discussion of this distinction, and see Brady (1994) for some discussion in the context of relevant logics. The logics in this paper are in the framework FMLA of Humberstone (2011), so the distinction between types of rules does not arise, but it would arise if one considered consequence relations, or SET-FMLA logics.

- $A \Rightarrow \Box A$ (Nec)
- $A \rightarrow B \Rightarrow \Box A \rightarrow \Box B$ (Mono)

This axiomatization is redundant, but the redundancies will facilitate the later discussion. So, I will leave them in. It is worth noting that given the other axioms and rules, (B) and (5) are interderivable. In terms of frames, the axioms (T), (4), (B), and (5) all correspond to S-based conditions on a frame $\langle W, S\rangle$, in the sense that for a given frame $\langle W, S\rangle$, S obeys the condition if and only if the axiom is valid on the frame. (T) corresponds to reflexivity, (4) to transitivity, and (B) to symmetry. (5) corresponds to a condition known as euclideanness, which is equivalent to symmetry given reflexivity and transitivity.

The epistemic logic S5 represents an idealization, since the knowers whose knowledge it codifies have unlimited memory and inferential capacities. While the logic S5 and its frames are perahps the starting point for epistemic logics, it does not have many defenders as it seems to represent overly idealized agents. Stalnaker (2006) has offered a limited defense of this idealization, although he thinks the modeling has serious flaws. Yap (2014) has defended idealizations in epistemic logic, generally and not just for the specific case of interest.

The logic S5 has been criticized in many different ways as an epistemic logic.[5] I will focus on criticisms directed at the (B) and (5) axioms, as these two principles are interderivable given the background principles. Hintikka has argued that the idealizations inherent in an S5 approach to knowledge are implausible because "[i]t is not true that everybody could come to know the possibility of any fact whatsoever simply by following the consequences of what he already knows."[6] Stalnaker (2006, 173) regards this objection as clear and decisive. As Hintikka notes, the mere fact that p is true need not imply that one knows that p is consistent with one's knowledge, where '$\neg\Box\neg p$' formalizes the claim that p is consistent with one's knowledge. There is an apparent gap between matters of non-epistemic fact and knowledge about one's epistemic states, so the (B) axiom must go, and along with it (5).

In the context of epistemic logics, (5) is often known as the *negative introspection* axiom, since not knowing something implies that one knows it is not known. It has come under much criticism as well. Hintikka (1962, 106) rejects it as implausible, as does Humberstone (1988, 187). The reason is that one might, when p is

[5] See Humberstone (2016, ch. 5) for an overview of objections in the context of an epistemic logic of knowledge and belief.

[6] Hintikka (1962, 54.)

false, mistakenly take oneself to know p, on the basis of misleading though persuasive evidence. The falsity of p would imply that one does not know p, and by (5), one knows that one does not know p. Given this knowledge, the agent should be able to see that they have made a mistake, even though they take their evidence to be conclusive. As Stalnaker (2006, 177) puts it, the (5) principle "require[s] that rational agents be immune to error." The possibility of error is important in epistemology, and plausibly epistemic logics should allow for that possibility.

The (5) axiom is inconsistent with so-called *Rumsfeld ignorance*, that there are unknown unknowns.[7] The inconsistency is immediate, since (5) says that if a claim is unknown, then one knows that it is unknown. It is plausible that the category of unknown unknowns is not incoherent, since it does seem that there are many things that are unknown and we are ignorant of that fact as well. Echoing Hintikka (1962, 106), "you may fail—unless you happen to be as sagacious as Socrates—to know your ignorance." The plausibility of the category of unknown unknowns provides additional reason to be suspicious of the (5) principle.

Further objections to the (5) principle arise in epistemic doxastic logics, which combine an operator for belief, $\mathcal{B}$, with the epistemic operator for knowlege, $\Box$. In such logics, one adopts plausible bridge principles relating belief and knowledge, such as $\Box A \to \mathcal{B}A$ and $\mathcal{B}A \to \mathcal{B}\Box A$. Humberstone (2016, 373ff.) illustrates the problems that arise from (5) by presenting an argument due to Wolfgang Lenzen. Starting from the assumption that the agent has a false belief, represented as $\neg p \wedge \mathcal{B}p$, one can derive that the agent has contradictory beliefs, namely $\mathcal{B}\Box p \wedge \mathcal{B}\neg\Box p$. Depending on the logic of belief, this result may be disastrous or merely extremely unsavory. Of the principles involved in the derivation, (5) appears to be the least plausible. Given the other problems with (5), it seems to be the natural place to pin the blame.

Williamson (2000, 166-167) argues that there are important epistemic asymmetries that arise in knowledge. In certain skeptical scenarios there is a good case, where one isn't being deceived, and a bad case where one is. If one is in the good case, the bad case is not epistemically accessible, while if one is in the bad case, then the good case is epistemically accessible. This sort of consideration provides a reason for rejecting the symmetry condition on S. Rejecting the symmetry of S brings with it a rejection of axioms (B) and (5). I will note that Williamson offers a more general argument, the anti-luminosity argument, against principles of the form $\_ \to \Box\_$. Such an argument cuts against the (5) axiom, as well as the (4) axiom, but I will not have anything further to say on the

[7] See Humberstone (2016, 371), Fine (2018), and Fan (2023) for more on Rumsfeld ignorance.

luminosity issue.

The epistemic logics discussed so far have all used *classical* logic. There is increasing interest in the use of *non-classical* logics in epistemic logics. There are two ways one might go about using non-classical logics. One is to keep the base logic classical while using a non-classical logic within the scope of the epistemic operators. This approach is adopted by Fagin and Halpern (1987), Levesque and Lakemeyer (2001), Sedlár and Vigiani (2023), Ferenz (2023), and Standefer et al. (2023), among others. This approach attempts to combine the features of classical logic in the base language with non-classical logic providing closure conditions on the epistemic concepts. On this approach, within the scope of epistemic modal operators, one is restricted to using the non-classical logic so that the consequences of a formula within the scope of an epistemic modal may differ from the consequences outside the scope of any epistemic modal. The second approach is to use a non-classical base logic instead of classical logic and add an epistemic extension. This approach is adopted by Bílková et al. (2016), Bilková et al. (2010), Sedlár (2015, 2016), and Punčochář et al. (2023), among others. The approach pursued below falls in this second category. I will use the relevant logic R as the base logic and study some epistemic extensions of it. With that, let us turn to the background on R.

3 Relevant logics

Relevant logics are logics whose implication connective enforces a strong connection between antecedent and consequent.[8] The most famous relevant logic is perhaps the logic R of Anderson and Belnap (1975), and that is the logic on which I will focus for this paper.[9] Focusing on R will facilitate comparison with the results of Standefer (2023b), but most of the results will carry over to other relevant logics as well. Relevant logics have been studied axiomatically, algebraically, and frame-theoretically, and in this section I will present axioms for R as well as the ternary relational frames for it.

Definition 3.1 (Ternary relational frames)**.** *A basic ternary relational frame* F *is a quadruple* $\langle \mathsf{K}, \mathsf{N}, \mathsf{R}, ^* \rangle$, *where* $\mathsf{K} \neq \emptyset$, $\mathsf{N} \subseteq \mathsf{K}$, $\mathsf{R} \subseteq \mathsf{K}^3$, $^* : \mathsf{K} \mapsto \mathsf{K}$, *where*

(B1) $a \leq b =_{\mathrm{Df}} \exists x \in \mathsf{N},\ \mathsf{R}xab,$

[8]See Read (1988),Dunn and Restall (2002), Bimbó (2007), Mares (2020), Logan (2024) for overviews of relevant logics.

[9]See Mares (2004) for an extended defense of R.

(B2) $\leq$ *is a partial order,*

(B3) $a^{**} = a$,

(B4) $a \leq b$ *only if* $b^* \leq a^*$, *and*

(B5) *if* $d \leq a$, $e \leq b$, $c \leq f$, *and* $Rabc$, *then* $Rdef$.

A basic ternary relational frame is an R*-frame, or a ternary relational frame, iff it obeys the following conditions.*

(C1) $Rabc \Rightarrow Rac^*b^*$

(C2) $Rabc \Rightarrow Rbac$

(C3) $Rabcd \Rightarrow \exists x \in K(Rbxd$ *and* $Racx)$

(C4) $Rabc \Rightarrow \exists x \in K(Rabx$ *and* $Rxbc)$

Separately listing the conditions on the basic frames and on the R-frames can be useful for seeing how seldom the additional features of R-frames are invoked. Since we are only considering R-frames in this paper, we will not refer to any frames as 'basic ternary relational frames'.

Before proceeding, it will be worth commenting on a few aspects of the frames. First, the points in the set K are often viewed as being situations, which are partial and potentially contradictory, as opposed to worlds, which are maximally determinate and consistent. Worlds, as maximal situations, will play no role in the discussion of this paper, although they are used by some relevant logicians.[10] Second, given that understanding of the points, the relation $\leq$ is naturally understood as a kind of mereological containment relation. If $a \leq b$, then the situation b contains the situation a. There is no requirement that there be a greatest situation containing any pair of situations.

The Routley star, *, and its interpretation will play an important role in the epistemic logics, and I will return to it and its interpretation in the next section. The ternary relation, while important in the models, will not play a major role in the philosophical interpretation of the epistemic logics. Therefore, I will not take a stand on its interpretation here.[11]

Next, we define models and the verification relation.

[10] See Meyer and Mares (1993) for an example.

[11] See Beall et al. (2012), Restall (2005), Mares et al. (2010), and Tedder (2023, 2021) for discussion of interpretations of the ternary relation. For critical commentary on it, see Brady (2017).

Definition 3.2 (Model). *A model M is a pair $\langle F, V\rangle$ of an R-frame F, $\langle K, N, R, ^*\rangle$, and a valuation function $V : \mathsf{At} \mapsto \wp K$ such that if $a \in V(p)$ and $a \leq b$, then $b \in V(p)$. Such a model is built on F.*

The valuation function is extended to a verification relation on the whole language as follows.

- $a \Vdash p$ iff $a \in V(p)$
- $a \Vdash \neg B$ iff $a^* \nVdash B$
- $a \Vdash B \wedge C$ iff $a \Vdash B$ and $a \Vdash C$
- $a \Vdash B \vee C$ iff $a \Vdash B$ or $a \Vdash C$
- $a \Vdash B \rightarrow C$ iff for all $b, c \in K$, if $Rabc$ and $b \Vdash B$, then $c \Vdash C$

Given this definition of the verification relation, we can provide the standard definitions of counterexample and validity.

Definition 3.3 (Counterexample, validity). *A model M is a counterexample to a formula A iff for some $a \in N$, $a \nVdash A$. The point a is a counterexample point.*

A formula A is valid on a frame F iff no model built on F is a counterexample to A.

A formula A is valid over a class of frames $\mathfrak{C}$ iff A is valid on each frame $F \in \mathfrak{C}$.

Write $\models_{\mathfrak{C}} A$ when A is valid over the class of frames $\mathfrak{C}$ and write $\models_{\mathsf{R}} A$ when A is valid over the class of R-frames.

I will note, without proof, two important lemmas appealed to without comment.

Lemma 3.1 (Heredity). *For all A in the language, for any model M, if $a \Vdash A$ and $a \leq b$, then $b \Vdash A$*

Lemma 3.2 (Verification). *The following are equivalent.*

- *For all $a \in N$, $a \Vdash A \rightarrow B$.*
- *For all $b \in K$, if $b \Vdash A$, then $b \Vdash B$.*

Both of these lemmas extend to the language of the next section, once knowledge operators are introduced.

Next, I will provide an axiomatization of R. The logic R is the least set of formulas containing axioms (R1)–(R11) and closed under the rules (R12) and (R13).

(R1) $A \to A$

(R2) $A \wedge B \to A$, $A \wedge B \to B$

(R3) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

(R4) $A \to A \vee B$, $B \to A \vee B$

(R5) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

(R6) $A \wedge (B \vee C) \to (A \vee B) \wedge (A \vee C)$

(R7) $(A \to \neg B) \to (B \to \neg A)$

(R8) $\neg\neg A \to A$

(R9) $(A \to (A \to B)) \to (A \to B)$

(R10) $A \to ((A \to B) \to B)$

(R11) $(A \to B) \to ((B \to C) \to (A \to C))$

(R12) $A, A \to B \Rightarrow B$

(R13) $A, B \Rightarrow A \wedge B$

Proof is defined in the usual way, as a sequence of formulas each of which is an axiom or follows from earlier members of the sequence by rules. We will write $\vdash_R A$ to mean that A has a proof and so is a theorem of R.

Belnap's variable-sharing property is an important feature of relevant logics, and it is arguably what makes them *relevant* logics.

Definition 3.4 (Variable-sharing Property)**.** *A logic* L *has the variable-sharing property iff for all formulas* A *and* B*, if* $\vdash_L A \to B$*, then* A *and* B *share a propositional atom.*

The variable-sharing property is preserved downwards to sublogics, a feature that will become important later. One of the crucial features of R is that it enjoys Belnap's variable-sharing property.

Theorem 1. R *has the variable-sharing property, i.e. for all formulas* A *and* B*, if* $\vdash_R A \to B$*, then* A *and* B *share a propositional atom.*

Proof. See Anderson and Belnap (1975, 252-254) or Restall (2000, 184-185) for a proof using matrix methods. □

The variable-sharing property provides a minimal level of formal relevance for valid implications and is typically taken as a necessary condition on being a relevant logic.[12] If a logic does not enjoy the variable-sharing property, then it is, consequently, not a relevant logic. Classical logic is a paradigm example of a non-relevant logic, with theorems such as $p \to (q \to q)$. The addition

[12] See Standefer (202x) for a discussion of variable-sharing as a characterization of relevant logics.

of new connectives to a relevant logic may lead to violations of variable-sharing, as is the case with the universal modality considered by Standefer (2023b).[13] It is, therefore, important for the relevant logician to verify that the addition of a connective to a relevant logic does not lead to violations of variable-sharing. This issue will be taken up again in the next section.

The axiom system above is adequate for the frame semantics of this section in the sense that it is sound and complete.

Theorem 2. *For all formulas* A, $\vdash_{\mathsf{R}} A$ *iff* $\models_{\mathsf{R}} A$.

Proof. See Routley et al. (1982, ch. 4) for a proof. $\square$

That is enough background on the base logic, R. In the next section, I will define some epistemic extensions of R. Some of these will be defined axiomatically and some will be defined by classes of frames. Both approaches, however, build on the presentations of R in this section.

4 Relevant epistemic logics

Having defined the base relevant logic and its frames, we can turn to the epistemic extensions. The epistemic logic of classical partition frames in §2 is S5, and that would be the natural neighborhood to look for epistemic extensions of R. As shown by Standefer (2023b), there are distinct options for an S5-ish logic extending R, but I will initially focus on two. There is the logic RS5 obtained axiomatically and the logic Eq obtained by expanding the ternary relational frames with an equivalence relation. While the use of equivalence relations is common in epistemic logics over classical logic, they have been studied much less in the context of relevant logics.

To define RS5, we add to R the several axioms and rules, adjusting the definition of proof and theorem accordingly. Those principles are just the ones from section §2 used to axiomatize S5 in the classical setting: (Agg)] $(\Box A \land \Box B) \to \Box(A \land B)$, (K) $\Box(A \to B) \to (\Box A \to \Box B)$, (T) $\Box A \to A$, (4) $\Box A \to \Box\Box A$, (B) $A \to \Box\neg\Box\neg A$, (5) $\neg\Box A \to \Box\neg\Box A$, (Nec) $A \Rightarrow \Box A$, and (Mono) $A \to B \Rightarrow \Box A \to \Box B$. In many ways, the theorems of RS5 are close to those of S5, but the former are properly contained in the latter. Apart from the base logic R not being as strong as classical logic, there are properly modal theorems of S5 that

[13] See Standefer (2022b) for a discussion of the role of variable-sharing in adding new connectives to relevant logics.

are not theorems of RS5. Two of which that will appear later are $\Box(A \lor B) \to (\Box A \lor \neg\Box\neg B)$ and $\Box(\Box A \lor B) \to (\Box A \lor \Box B)$.

To provide frames for the modal logic RS5, we add a binary accessibility relation, S, to ternary relational frames.

Definition 4.1 (RS5-frame)**.** *An* RS5*-frame is a quintuple* $\langle K, N, R, S, ^* \rangle$*, where* $\langle K, N, R, ^* \rangle$ *a ternary relational frame and* $S \subseteq K^2$*, obeying the following conditions.*

(S1) *If* Sbc *and* $a \leq b$*, then* Sac.

(S2) *If* $a \in N$ *and* Sab*, then* $b \in N$. (Nec)

(S3) Saa. (T)

(S4) *If* Sab *and* Sbc*, then* Sac. (4)

(S5) *If* Sab*, then* Sb^*a^*. (B)

(S6) *If* $\exists z(Rabz$ *and* $Szc)$*, then* $\exists x \exists y(Sax, Sby,$ *and* $Rxyc)$. (K)

The definitions of model, counterexample, holding, and validity are all adapted in a straightforward way. The verification condition for $\Box$ is the following.

- $a \Vdash \Box B$ iff for all b such that Sab, $b \Vdash B$

Soundness and completeness of RS5 with respect to these frames was proved by Fuhrmann (1990). In the context of RS5-frames, the (B) axiom corresponds to the condition that if Sab then Sb^*a^*. This means that the S relation is not generally an equivalence relation. Given the other conditions, it turns out that there are stringent limits on what sort of equivalence relations are available for RS5-frames, because some equivalence relations invalidate the (Nec) principle, as we shall see.

Generally, the S relation in RS5 models does not represent indiscernibility, as it is not an equivalence relation. It does not have as neat an epistemic interpretation as one finds with the equivalence relations. Punčochář et al. (2023) say, in effect, that $\{b \in K : Sab\}$ is the epistemic state of the agent at the situation a, the set $\{b \in K : Sab\}$ is a particular body of information for the agent at that situation. This is a slightly different conception of knowledge, knowledge as what is supported by one's information state, than one finds with the equivalence relations. Despite this difference, RS5 has many pleasant features, such as the usual collapse of modalities, and it need not totally undermine the appeal of RS5 as a logic for knowledge. Rather, the RS5-models are modeling a different conception of knowlecge than the classical partition models. It is unfortunate to give up on the intuitive picture offered by the use of equivalence relations to represent indistinguishability, so I will turn to the logic of the equivalence conception of knowledge, Eq, to be defined below.

Definition 4.2 (Equivalence frame)**.** *An equivalence frame is a quintuple* $\langle K, N, R, \approx, ^* \rangle$*, where* $\langle K, N, R, ^* \rangle$ *is a ternary relational frame and* $\approx$ *is an equivalence relation on* K*, obeying the condition that if* $a \leq b$ *and* $b \approx c$*, then there is* $x \in K$ *such that* $x \leq c$ *and* $a \approx x$*.*

The class of equivalence frames will be denoted $\mathfrak{Eq}$*.*

Validity is adapted in the obvious way, and we will write $\models_{\mathfrak{Eq}} A$ when A is valid over $\mathfrak{Eq}$.

This definition of equivalence frame differs from that of Standefer (2023b) in the interaction condition between $\leq$ and $\approx$. The condition of Standefer (2023b) is that if $a \leq b$ and $b \approx c$, then $a \approx c$. This is the interaction condition used by RS5-frames. The condition adopted in definition 4.2 is weaker than the condition for RS5-frames, and it is also more plausible from an epistemic point of view.[14] If two points, b and c are indistinguishable, namely $b \approx c$, and b contains a, $a \leq b$, generally one would not expect a and c to be indistinguishable. Perhaps c contains a lot of things that are entirely absent from a. For example, suppose that situation b involves an apartment, α_1, with two rooms, a bedroom and a kitchen, while situation c involves a distinct but indistinguishable apartment, α_2, with two rooms, a bedroom and a kitchen. Then, situation a will involve just the kitchen of α_1. While b and c are indistinguishable, it does not follow that a and c are, since c contains a bed while a does not. Rather, there is a part of c, which we can call d, that involves only the bedroom of α_2, and, plausibly, a and d are indistinguishable. Therefore, I take the interaction condition to be motivated.

The verification condition for $\Box$, presented using $\approx$, is the following.

- $a \Vdash \Box B$ iff for all b such that $a \approx b$, $b \Vdash B$

With this in hand, we can define the logic Eq.

Definition 4.3 (The logic Eq)**.** *The logic* Eq *is the set of formulas valid on all equivalence frames, i.e. the set* $\{A \in \mathcal{L} : \models_{\mathfrak{Eq}} A\}$*.*

[14] We could correspondingly weaken the condition on RS5-frames without changing the logic.

As shown by Standefer (2023b), the following principles are invalid on the class of equivalence frames.

- (K) $\Box(A \to B) \to (\Box A \to \Box B)$
- (B) $A \to \Box\neg\Box\neg A$
- (5) $\neg\Box A \to \Box\neg\Box A$
- (Nec) $A \Rightarrow \Box A$

I will demonstrate the invalidity of (B), (5), and (Nec) here. I will begin by demonstrating that (5) entails (B), given some additional background. While this is familiar from classical S5, it is worth reproducing in this non-classical context.[15]

Lemma 4.1. *In any logic extending* R *with* (T), (5) *implies* (B).

Proof. From (5), we get $\neg\Box\neg\Box\neg A \to \Box\neg A$ via contraposition and double negation elimination. Using (T) and transitivity, we have $\neg\Box\neg\Box\neg A \to \neg A$. Contraposing again gets $A \to \Box\neg\Box\neg A$, as desired. $\Box$

To show the invalidity of (B), I will use the frame in table 1, where $K = \{0, a, b\}$ and $N = \{0\}$. To read the table, the first column is the first position of the ternary relation, the top row is the second, and the third position is represented by each point in the middle cells. For example, the upper cell containing '$0ab$' should be understood as saying that $Rab0$, $Raba$, and $Rabb$. Similarly, the $\approx$ row for 0 means that $0 \approx 0$ and $0 \approx b$.

R	0	a	b	*	≈
0	0	a	b	0	0, b
a	a	a	0ab	b	a
b	b	0ab	b	a	0, b

Table 1: Equivalence frame F_1

Lemma 4.2. *The frame* F_1 *without the equivalence relation in table 1 is a ternary relational frame. Further,* F_1 *with the equivalence relation is an equivalence frame.*

[15] Since R is the only relevant logic discussed in this paper, the lemma concerns R. Inspection of the proof reveals that it works for much weaker logics. Additionally, the converse entailment, from (B) to (5) holds as well, but it is not needed for the following results.

Proof. The proof is by inspection of all the conditions, which will be omitted here. $\square$

Lemma 4.3. (B) *is invalid over the class* $\mathfrak{Eq}$.

Proof. Let $\nu(p) = \{a, 0\}$. Then, $a \Vdash \neg p$, so $a \Vdash \Box\neg p$. It then follows that $b \nVdash \neg\Box\neg p$. Therefore, $0 \nVdash \Box\neg\Box\neg p$, As $0 \Vdash p$, this is sufficient to invalidate $p \to \Box\neg\Box\neg p$. $\square$

Corollary 4.1. (5) *is invalid over the class* $\mathfrak{Eq}$.

Proof. This follows from the previous lemma and the fact that (T) is valid. $\square$

Next we turn to (Nec).

Lemma 4.4. *The rule* (Nec) *fails for* Eq.

Proof. Let M be a ternary relational model where for some point $a \in K$, $a \nVdash p \to p$. We know there is such a model since $q \to (p \to p)$ is not a theorem of R. Add to this model an equivalence relation $\approx$ where for all $a, b \in K$, $a \approx b$. Therefore, for some $b \in N$, $b \nVdash \Box(p \to p)$, as desired. $\square$

While many of the typical modal principles of S5 fail for Eq, some are valid, namely (4), (T), and (Mono).

An important question was left open by Standefer (2023b): Does Eq satisfy the variable-sharing criterion? This is an important question because being a proper relevant logic requires an affirmative answer. I can provide an affirmative answer to the former, namely that for all A and B, if $A \to B$ is a theorem of Eq, then A and B share an atom. To prove this, I will use the frame in table 2, where $K = \{0, a, b\}$ and $N = \{0\}$.

R	0	a	b	*	≈
0	0	a	b	0	0
a	a	a	0ab	b	a
b	b	0ab	b	a	b

Table 2: Equivalence frame F_2

Lemma 4.5. *The frame* F_2 *without the equivalence relation in table 2 is a ternary relational frame. Further,* F_2 *with the equivalence relation is an equivalence frame.*

Proof. The underlying ternary relational frame is the same as the frame in table 2. □

Lemma 4.6. *Let* A *and* B *be formulas with no variables in common. Over the frame* F_2*, define a model as follows. For all atoms* p *occurring in* A*, let* $V(p) = \{a\}$*. For all atoms* q *occurring in* B*, let* $V(q) = \{b\}$*. The following are true.*

(i) For all formulas C *only containing atoms occurring in* A*,* $a \Vdash C$*.*

(ii) For all formulas C *only containing atoms occurring in* A*,* $b \nVdash C$*.*

(iii) For all formulas D *only containing atoms occurring in* B*,* $b \Vdash D$*.*

(iv) For all formulas D *only containing atoms occurring in* B*,* $a \nVdash D$*.*

Proof. For this proof, C will be used for formulas only containing atoms occurring in A and D will be used for formulas only containing atoms occurring in B. The proof is by simultaneous induction on C and D. For atoms, the four cases are immediate.

The cases for conjunction and disjunction are immediate by the inductive hypothesis.

Suppose C is $\neg E$. By the inductive hypothesis, $b \nVdash E$, so $a \Vdash \neg E$. By the inductive hypothesis again, $a \Vdash E$, so $b \nVdash \neg E$. The cases for D are similar.

Suppose C is $E \to F$. Suppose that $a \nVdash E \to F$, so for some $x, y \in K$, $Raxy$, $x \Vdash E$ and $y \nVdash F$. It follows from the inductive hypothesis that $x \in \{a, 0\}$. If $x = 0$, then $y = a$. In that case, by the inductive hypothesis, $y \Vdash F$, which is a contradiction. If $x = a$, then $y = a$, so there is a contradiction by the inductive hypothesis.

Next, suppose that $b \Vdash E \to F$. By the inductive hypothesis, $a \Vdash E$. Since $Rbab$, $b \Vdash F$. By the inductive hypothesis, however, $b \nVdash F$, which is a contradiction. Therefore, $b \nVdash E \to F$.

The remaining conditional cases are handled similarly.

The cases where C is $\Box E$ are immediate from the inductive hypothesis and the fact that each equivalence class is a singleton.

□

Lemma 4.7. *If* A *and* B *share no atoms, then* $A \to B$ *has a counterexample on* F_2*.*

Proof. Over the frame F_2, let V be a valuation such that

- for all atoms p occurring in A, $V(p) = \{a\}$, and

- for all atoms q occurring in B, $V(q) = \{b\}$.

By lemma 4.6, $a \Vdash A$ but $a \nVdash B$. Therefore $A \to B$ does not hold in this model. □

With those pieces in hand, I can prove the variable-sharing theorem.[16]

Theorem 3 (Variable-sharing for Eq)**.** *Suppose that* $A \to B$ *is a theorem of* Eq*. Then* $A \to B$ *share an atomic formula.*

Proof. Suppose A and B do not share a variable. By lemma 4.7, it has a counterexample on an equivalence frame. Therefore, $A \to B$ is not a theorem of Eq. □

Theorem 3 settles one of the questions left open by Standefer (2023b). The logic Eq enjoys variable-sharing and is a reasonable candidate for an epistemic relevant logic. The question of an axiomatization is still left open, but an axiomatization is not needed to demonstrate variable-sharing using the technique above.

The logic Eq has some nice features that make it appealing as an epistemic logic. First, its frames use equivalence relations, so they appeal to the same sort of intuitions about indiscernibility that we find with the classical partition frames. They have a clear intuitive interpretation, as opposed to the situation with the accessibility relation S in the RS5 frames. The intuitions behind the equivalence conception of knowledge are worth preserving, at least as a starting point, since they are so robust and so useful, as demonstrate by the logical work exploring extensions of it.

Second, many of the criticisms of S5 as an epistemic logic focus on the (B) and (5) axioms. Even in very weak relevant logics, these axioms are interderivable, given the other modal principles, so they stand for fall together here as well. Neither of these axioms is valid over Eq. Since relevant logics are paraconsistent, one could have unknown unknowns along with the (5) axiom without the logic collapsing into triviality. That would not be ideal, since the concept of unknown unknowns is not on its face incoherent. The logic Eq allows there to be unknown unknowns consistently. Similarly, the mere truth of a claim does not on its own imply that an agent knows the truth of that claim is consistent with their knowledge. The logic Eq does not build in unrestricted negative introspection in the way that S5 and RS5 do. Thus, the equivalence frames, and the logic

[16] It is worth noting that Brady et al. (2003, 100ff.) demonstrate a correspondence between the frame F_2 Belnap's 8-valued matrix M_0 used in Anderson and Belnap (1975) to show that R and E enjoy variable-sharing.

they generate, avoid some of the major criticisms leveled against their classical counterparts.

The preceding considerations prompt the question of what sort of epistemic agent Eq is supposed to model. The agents of Eq are highly idealized, since they have no limits on memory or computational resources. They have unlimited positive introspection, as the (4) axiom is valid. Their powers of introspection are, nonetheless limited in certain ways. The failure of the (5) axiom means that they are unaware of their own ignorance, and one can consistently model situations in which they do not know what they do not know. Similarly, the failure of the (B) axiom means that they are limited in what consequences they draw from their own knowledge. The fact that p is true need not imply that an agent know that p is consistent with their knowledge, since they may be mistaken and their knowledge is already inconsistent. The agents modeled by the equivalence frames of Eq are in some respects highly idealized, but in other respects they are rather limited. This combination is, I think, potentially fruitful for further exploration of epistemic concepts.

The source of the failure of the (B) and (5) axioms is due to the Routley star, the part of the frame that interprets negation.[17] While we may have two points, a and b, in the same equivalence class, there is no guarantee that their stars, a^* and b^*, will be in the same class. The star does not respect the equivalence classes.[18]

[17] Rather than using the Routley star, one could use a binary compatibility relation to interpret negation in the ternary relational models, following Dunn (1993) and Restall (1999), among others. Compatibility generalizes the Routley star, and many of the same points made below will carry over. Therefore, I will not further consider the use of compatibility relations.

One might also follow Dunn (1995) in wanting to consider what happens when one has a primitive possibility operator, $\Diamond$, not defined in terms of negation and necessity. The short answer is that it will depend on whether $\Box$ and $\Diamond$ are interpreted using distinct accessibility relations or not. If they are, then the resulting logic will be similar to that of the coordinated equivalence frames defined below. If they are not, which is the standard default option in this area, then the logic will be similar to that of the equivalence frames, with the same axioms and being invalidated.

[18] This issue is arising because I am following the Australian plan for negation in relevant logics, as defended by Berto (2015) and Berto and Restall (2019). An alternative is the American plan, recently defended by De and Omori (2018). On the American plan, negation is not interpreted using the star or a compatibility relation. Instead, one has a verification relation, $\Vdash^+$ and a falsification relation, $\Vdash^-$. One then has the conditions

- $a \Vdash^+ \neg B$ iff $a \Vdash^- B$, and
- $a \Vdash^- \neg B$ iff $a \Vdash^+ B$.

If one adopts this approach, no analog of the logic Eq will arise, as (B) and (5) will be valid. This suggests that further consideration of both the Australian and American plans in the context of

If, following Restall (1999), we think of b^* as the point that is maximally compatible with b, it is plausible that one may be unable to distinguish situations a and b while being able to distinguish a^* and b^*. While one's information does not permit one to distinguish a and b, the information that the agent is lacking may obscure important differences in the information of those two points. The maximal point compatible with a may then be distinguishable from the maximal point compatible with b, which is to say that $a \in [b]$ need not imply $a^* \in [b^*]$.

While it is plausible to suppose that the star need not respect equivalence relations, from a logical point of view it is worth considering the frames that result from the assumption that the star does respect equivalence classes. This would be the assumption that when two situations are indiscernible to an agent, the situations maximally compatible with them are also indiscernible. For this assumption to be plausible, agents would need more awareness of the information omitted from a situation, since that is salient to determining what maximally compatible situations. Intuitively, this would be an increased idealization of the agents' powers of negative introspection, but in order to make good on this idea, let us define the frames at issue.

Definition 4.4 (Coordinated equivalence frames)**.** *An equivalence frame* $\langle K, N, R, \approx, ^* \rangle$ *is a coordinated equivalence frame iff for all* $a, b \in K$, $[a] = [b]$ *implies* $[a^*] = [b^*]$.

Let $\mathfrak{CEq}$ *be the class of all coordinated equivalence frames.*

We will write $\models_{\mathfrak{CEq}} A$ *when* A *is valid over* $\mathfrak{CEq}$.

Definition 4.5 (The logic CEq)**.** *Let* CEq *the set of all formulas valid over* $\mathfrak{CEq}$, *i.e. the set* $\{A \in \mathcal{L} : \models_{\mathfrak{CEq}} A\}$.

Clearly, $\mathsf{Eq} \subseteq \mathsf{CEq}$. In fact, this containment is proper, as the logic of coordinated equivalence frames is much stronger than that of equivalence frames.

Theorem 4. *The following are valid over coordinated equivalence frames*

(i) (B) $A \to \Box\neg\Box\neg A$

(ii) (5) $\neg\Box A \to \Box\neg\Box A$

(iii) (MC) $\Box(A \vee B) \to (\Box A \vee \neg\Box\neg B)$

Proof. We will prove that (5) is valid, leaving the rest to the reader. Suppose that $a \Vdash \neg\Box A$ but $a \nVdash \Box\neg\Box A$. Then for some $b \in [a]$, $b \nVdash \neg\Box A$. Therefore,

modal extensions of relevant logics would be worthwhile.

$b^* \Vdash \Box A$, so for all $c \in [b^*]$, $c \Vdash A$. As $a \Vdash \neg\Box A$, it follows that $a^* \nVdash \Box A$. Since $b \in [a]$, it follows that $[b^*] = [a^*]$. With $a^* \nVdash \Box A$, this implies that for some $c \in [b^*]$, $c \nVdash A$, which is a contradiction. Therefore, (5) is valid.

$\Box$

Corollary 4.2. Eq *is properly contained in* CEq.

There is still some distance from the RS5-frames with which we started, but we can make a connection to another class of frames, relatives of some frames studied by Mares (1994).

Definition 4.6 (Mares-Meyer modal frame)**.** *A Mares-Meyer modal frame is a quintuple* $\langle K, N, R, S, ^* \rangle$*, where* $\langle K, N, R, ^* \rangle$ *a ternary relational frame and* $S \subseteq K^2$*, obeying the following conditions.*

(MM1) *If* Sbc *and* $a \leq b$*, then* Sac.

(MM2) *If* Sab*, then* $\exists x(x \leq b, Sax,$ *and* $Sa^*x^*)$

(MM3) Saa.

(MM4) *If* Sab *and* Sbc*, then* Sac.

(MM5) *If* Sab*, then* Sb^*a^*.

We can define a new relation, T, on a frame as follows.

- Tab iff both Sab and Sa^*b^*.

Using this definition, (MM2) can be written as if Sab, then $\exists x(x \leq b$ and $Tax)$. These frames are related to another class of frames that will be of interest to us, Mares's mostly Meyer modal models.

Definition 4.7 (MMR-frames)**.** *An* MMR*-frame is a quintuple* $\langle K, N, R, S, ^* \rangle$*, where* $\langle K, N, R, ^* \rangle$ *an* R*-frame and* $S \subseteq K^2$*, obeying the following conditions.*

(M1) *If* Sbc *and* $a \leq b$*, then* $\exists x(Sax$ *and* $x \leq c)$.

(M2) *If* Sab*, then* Sa^*b^*

(M3) Saa.

(M4) *If* Sab *and* Sbc*, then* Sac.

(M5) If Sab*, then* Sb^*a^**.*

These are of interest for the following reason.

Theorem 5. MMR*-frames are coordinated equivalence frames.*

Proof. First we show that S obeys symmetry. Suppose Sab. By (M2), Sa^*b^*. By (M6), $Sb^{**}a^{**}$. As $c^{**} = c$, for all $c \in K$, Sba. Since S obeys reflexivity and transitivity, it follows that S is an equivalence relation.

Next, we show that if $[a] = [b]$, then $[a^*] = [b^*]$. Suppose $[a] = [b]$, so Sab. By (M6), Sb^*a^*, so $[a^*] = [b^*]$. □

Using the work of Mares (1994), we have the following theorem.

Theorem 6. *Given a Mares-Meyer modal frame* $\langle K, N, R, S,^* \rangle$*,* $\langle K, N, R, T,^* \rangle$ *is a* MMR-*frame.*

Proof. The proof follows that of section 7 of Mares (1994). The results extend to the frames obeying postulates (M3)–(M5), as noted by Mares. □

Given a Mares-Meyer modal frame $\langle K, N, R, S,^* \rangle$, one can construct an associated coordinated equivalence frame, $\langle K, N, R, T,^* \rangle$ using the previous theorem. The addition of a valuation results in them agreeing pointwise on formulas.

Theorem 7. *Let* $\langle K, N, R, S,^* \rangle$ *be a Mares-Meyer modal frame and* $\langle K, N, R, T,^* \rangle$ *the constructed coordinated equivalence frame of theorem 6. Let* M *be the model obtained by adding a valuation* V *to the Mares-Meyer modal frame and let* N *be the model obtained by adding* V *to the constructed coordinated equivalence frame. Then, for all* $a \in K$ *and all formulas* A*, in* M*,* $a \Vdash A$ *iff in* N*,* $a \Vdash A$*.*

Proof. The proof is by induction on the construction of A. See Mares (1994) lemma 7.3 for details. □

Let HCEq be the logic R+{ (Agg), (Mono), (T), (4), (B), (5), (MC)}. The logic HCEq is sound with respect to the class of Mares-Meyer modal frames.

Theorem 8. *If* A *is a theorem of* HCEq*, then* A *is valid over the class of Mares-Meyer modal frame.*

Proof. The proof is by induction on the construction of proof in HCEq. Condition (MM2) yields the validity of (MC). The rest are straightforward. □

Corollary 4.3. *If* A *is a theorem of* HCEq*, then* A *is a theorem of* CEq*.*

Finally, we come to Completeness.

Theorem 9. *If* A *is valid over the class of Mares-Meyer modal frames, then* A *is a theorem of* HCEq.

Proof. The proof is by a standard canonical model construction. See Mares (1992, 1993), Restall (2000, ch. 11), or Standefer (2020) for examples. □

Corollary 4.4. *If* A *is a theorem of* CEq, *then* A *is a theorem of* HCEq.

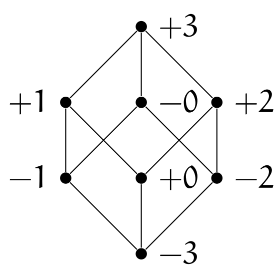
Proof. Given a non-theorem A of HCEq, we can construct a counterexample Mares-Meyer modal model. Using theorems 6 and 7, we can construct an associated co-ordinated equivalence frame that also serves as a counterexample. □

Corollary 4.5. HCEq=CEq.

Given the axioms of HCEq, we can adapt the proof of variable-sharing due to Belnap to show that CEq has the variable-sharing property.[19]

Theorem 10 (Variable-sharing for CEq)**.** CEq *enjoys the variable-sharing property.*

Proof. We will use a matrix-based argument. The matrix M_0 is presented in table 3. In this matrix, the set of designated values are the ones marked with $+$. A val-



$\to$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$	$\neg$
-3	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$	$+3$
-2	-3	$+2$	-3	$+2$	-3	-3	$+2$	$+3$	$+2$
-1	-3	-3	$+1$	$+1$	-3	$+1$	-3	$+3$	$+1$
-0	-3	-3	-3	$+0$	-3	-3	-3	$+3$	$+0$
$+0$	-3	-2	-1	-0	$+0$	$+1$	$+2$	$+3$	-0
$+1$	-3	-3	-1	-1	-3	$+1$	-3	$+3$	-1
$+2$	-3	-2	-3	-2	-3	-3	$+2$	$+3$	-2
$+3$	-3	-3	-3	-3	-3	-3	-3	$+3$	-3

Table 3: Matrix M_0 for variable sharing

uation v assigns atoms values and the values of complex formulas are computed

[19] See Anderson and Belnap (1975, 252-254). This proof was similarly adapted for use with a relevant modal logic by Standefer (2023b). Robles and Méndez (2011; 2012) for a general characterization of matrices that can be used to demonstrate variable-sharing.

using the tables. For the modals, we will use the condition $v(\Box A) = v(A)$. The validity of a formula on this matrix is defined as having a designated value on all valuations. All the axioms of R are valid and the rules preserve validity. Further, the modal axioms of CEq are valid and the rule also preserves validity, therefore CEq is sound for this matrix.

The sets $\{+1, -1\}$ and $\{+2, -2\}$ are both closed under all the operations corresponding to the connectives. Therefore, by induction, if for all atoms p in a formula A, $v(p) \in \{+1, -1\}$, then $v(A) \in \{+1, -1\}$, and similarly for $\{+2, -2\}$. If $v(A) \in \{+1, -1\}$ and $v(B) \in \{+2, -2\}$, then $v(A \to B)$ is not designated. If A and B are assumed to have no atoms in common, then we can construct an assignment v such that $v(A) \in \{+1, -1\}$ and $v(B) \in \{+2, -2\}$, which then suffices for $v(A \to B)$ not being designated. Therefore, $A \to B$ is not a theorem of CEq. □

As a corollary, this gives an alternative proof that Eq has the variable-sharing property.

Corollary 4.6. Eq *enjoys the variable-sharing property.*

Proof. CEq enjoys the variable-sharing property. As Eq is a sublogic of CEq and the variable-sharing property is preserved to sublogics, Eq enjoys it as well. □

The introduction of Mares-Meyer modal frames and MMR-frames is in part a technical device to permit an axiomatization and proof of Completeness for CEq. Nonetheless, we can make a few comments on the Mares-Meyer modal frames.

The accessibility relation of the Mares-Meyer modal frames gives a partial description of indistinguishability, in the sense that one can define an equivalence relation from it. The description is partial in the sense that Sab on its own does not say that the agent cannot distinguish a and b, but instead requires the addition of Sa^*b^* to conclude that the agent cannot distinguish a and b.

Next, we will comment on the relation between the class of Mares-Meyer modal frames and the class of RS5-frames. Both are subclasses of a more general class, namely the frames obeying the postulates.

- If Sbc and $a \leq b$, then Sac.
- Saa.
- If Sab and Sbc, then Sac.
- If Sab, then Sb^*a^*.

These are frames for the logic R.T4B, which includes (Mono) and (Agg). The RS5-frames add additional postulates to ensure the validity of (K) and (Nec), while the Mares-Meyer modal frames add a postulate for the validity of (MC). One could add (MC) to RS5, adding the corresponding postulate (MM2) to the RS5-frames to maintain Soundness and Completeness. Similarly, one could add postulates for (K) and (Nec) to the Mares-Meyer modal frames, which would transfer to the coordinated equivalence frames, by a result of Mares (1994). I have not added the extra conditions, because they would greatly restrict the equivalence relations available on the coordinated equivalence frames, which would undermine the intuitive interpretation of the equivalence relations as encoding indistinguishability.

Despite these connections, one should not think that Eq and CEq are sublogics of RS5. There is a theorem of Eq that is not a theorem of RS5: $\Box(\Box A \lor B) \to (\Box A \lor \Box B)$.[20]

Theorem 11. *The formula* $\Box(\Box A \lor B) \to (\Box A \lor \Box B)$ *is a theorem of* Eq.

Proof. Let M be an equivalence model. Suppose $a \Vdash \Box(\Box A \lor B)$. Suppose that $a \nVdash \Box A \lor \Box B$. It follows that $a \nVdash \Box A$ and $a \nVdash \Box B$. It follows that there are $b, c \in [a]$ such that $b \nVdash A$ and $c \nVdash B$. From the initial assumption, it follows that for all $d \in [a]$, $d \Vdash \Box A \lor B$, so $c \Vdash \Box A \lor B$. This implies $c \Vdash \Box A$, which in turn implies that for all $e \in [c]$, $e \Vdash A$. As $[c] = [a]$, it follows that $b \Vdash A$, which is a contradiction. Therefore, $a \Vdash \Box A \lor \Box B$, as desired.

$\Box$

There are instances of the scheme $\Box(\Box A \lor B) \to (\Box A \lor \Box B)$ that are not theorems of RS5.

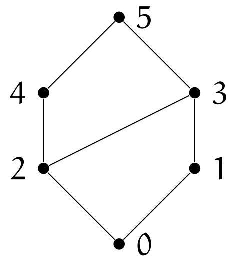
Theorem 12. $\Box(\Box p \lor q) \to (\Box p \lor \Box q)$ *is not a theorem of* RS5.

Proof. The matrix in table 4 was found by MaGIC provides to provide a counterexample.[21] The set of designated elements is $\{2, 3, 4, 5\}$. A counterexample is any valuation that assigns a formula a value outside of the set of designated values.

To obtain the counterexample, we use a valuation v assigning $v(p) = 2$ and $v(q) = 1$. It follows that $v(\Box(\Box p \lor q)) = 3$ and $v(\Box p \lor \Box q) = 2$, which suffices for $v(\Box(\Box p \lor q) \to (\Box p \lor \Box q)) = 0$. As all the axioms of RS5 are designated

[20] This formula is discussed by Ono (1977) in the context of intuitionistic versions of S5.

[21] John Slaney's program MaGIC can be found at `http://users.cecs.anu.edu.au/~jks/magic.html`.



$\to$	0	1	2	3	4	5	$\neg$	$\Box$
0	5	5	5	5	5	5	5	0
1	0	2	0	4	0	5	4	0
2	0	1	2	3	4	5	3	2
3	0	0	0	2	0	5	2	3
4	0	0	0	1	2	5	1	2
5	0	0	0	0	0	5	0	5

Table 4: Six-element MaGIC counterexample

on this matrix and the rules preserve designation, it follows that $\Box(\Box p \vee q) \to (\Box p \vee \Box q)$ is not a theorem. □

There are theorems of RS5 that are not theorems of Eq, an example of which we can identify with an application of the rule (Nec).

Lemma 4.8. *The formula* $\Box(p \to p)$ *is not a theorem of* CEq.

Proof. This is proved by noting that the frame used in the proof of lemma 4.4 is, in fact, a coordinated equivalence frame. □

Corollary 4.7. *The logic* RS5 *is incomparable with* Eq *and with* CEq.

While RS5 and the equivalence logics are incomparable, it is worth noting a logic that extends them all. Let RS5MC be the logic RS5 extended with (MC).

Theorem 13. *The logic* RS5MC *properly extends* CEq.

Proof. All the axioms of HCEq are axioms of RS5MC, and they share the same rules. The containment is proper because the rule (Nec) permits the derivation of $\Box(p \to p)$, which is not a theorem of CEq by lemma 4.8. □

The relationships between four modal logics discussed in this paper are summarized in table 5. As a complete axiomatization of Eq is not yet known, so the axioms and rules listed here should not be taken as exhaustive. Let us turn to some concluding discussion of these epistemic logics.

Name	Modal principle	Eq	CEq	RS5	RS5MC
(Agg)	$(\Box A \wedge \Box B) \to \Box(A \wedge B)$	✓	✓	✓	✓
(K)	$\Box(A \to B) \to (\Box A \to \Box B)$			✓	✓
(T)	$\Box A \to A$	✓	✓	✓	✓
(4)	$\Box A \to \Box\Box A$	✓	✓	✓	✓
(B)	$A \to \Box\neg\Box\neg A$		✓	✓	✓
(5)	$\neg\Box A \to \Box\neg\Box A$		✓	✓	✓
(MC)	$\Box(A \vee B) \to (\Box A \vee \neg\Box\neg B)$		✓		✓
	$\Box(\Box A \vee B) \to (\Box A \vee \Box B)$	✓	✓		✓
(Nec)	$A \Rightarrow \Box A$			✓	✓
(Mono)	$A \to B \Rightarrow \Box A \to \Box B$	✓	✓	✓	✓

Table 5: Modal principles of the different logics

5 Discussion

So far, I have introduced three main epistemic relevant logics, RS5, Eq, and CEq, with a fourth logic, RS5MC, introduced as a logic extending all of them. Both equivalence frames and coordinated equivalence frames utilize the indistinguishability approach to knowledge. The coordinated equivalence frames validate some of the more controversial principles, (B) and (5). These are not valid over the class equivalence frames.

From the point of view of the objections to S5 raised in section 2, Eq appears to be the clear winner, avoiding (B) and (5), as well as (Nec). None of the logics canvassed avoid the Williamsonian objections to positive introspection, (4).[22] Indeed, it appears difficult to avoid (4) while using frames with equivalence relations. If one is less moved by the objections to negative introspection, perhaps the more S5-like logics, such as CEq or RS5MC would be more appealing.

While in some ways similar to S5, these epistemic logics offer distinctive responses to issues of logical omniscience. Logical omniscience comprises a cluster of issues regarding ways in which an agent's knowledge is closed under various logical operations. The issues do not all stand or fall together in the current set-

[22] It is worth noting that one can, potentially, hold on to (Nec) while avoiding the Williamsonian objections. The reason is that in the present setting, the validity of $\Box A$ does not generally imply $A \to \Box A$. There are plausibly other reasons to avoid (Nec), discussed in this section, but it may be worth investigating how strong a logic with (Nec) can be while avoiding the Williamsonian objection.

ting.

Both the classes of equivalence frames and of coordinated equivalence frames invalidate (K) and (Nec). As a consequence, the sorts of agents they model do not automatically know all logical truths. Further, knowledge is not closed under implications in one sense, namely the sense of the (K) axiom $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$. There are related senses in which knowledge is closed, such as $(\Box(A \rightarrow B) \wedge \Box A) \rightarrow \Box B$, which is is valid.[23] Since (Mono) is built in to this approach to epistemic relevant logics, knowledge will be closed under *logically valid* implications, although not *merely assumed* implications. As a consequence of closure under (Mono), if an agent knows anything, they will know claims of arbitrarily high complexity, measured in terms of the number of logical connectives. Further, agents will know things involving arbitrarily many propositional variables. Thus, there is no sort of limitations in place on the basis of concepts or subject matters involved.[24]

Even though knowledge in these epistemic logics is closed under valid implications, this does not mean that agents know all logical truths. In fact, they can know some logical truths without knowing all of them. Further, they can know some classical tautologies without knowing all classical tautologies. Knowledge being closed under valid implications does not have as large negative effects as the knowledge of S5 being closed under valid implications. Even when knowledge is closed under valid relevant implications, there are still many logical truths that can remain unknown to the agents. In fact, agents can remain wholly ignorant of logical truths.

Adapting definitions from Williamson (2006), one can understand a formula context C to be *hyperintensional* iff there are formulas A and B for which the implication $\Box(A \leftrightarrow B) \rightarrow \Box(C(A) \leftrightarrow C(B))$ is not valid, where $A \leftrightarrow B$ is defined as $(A \rightarrow B) \wedge (B \rightarrow A)$. Perhaps surprisingly, knowledge operators in all the epistemic relevant logics under consideration in the previous section exhibit a degree a hyperintensionality, which follows from the results of Standefer (2023a).[25], This means that equivalences that are assumed known will not always

[23] Most of the features of the epistemic logics discussed in this paper do not depend on the choice of R as the base logic. In fact, the results carry over to all the other standard relevant logics. One exception is the validity of $(\Box(A \rightarrow B) \wedge \Box A) \rightarrow \Box B$, which does depend on the choice of R. If this seems like too much of a cost, it can be avoided by working with a weaker base logic.

[24] See Hawke et al. (2020) for discussion of subject matter and knowledge, and see Ferguson (2017) for an overview of conceptivist logics.

[25] See Berto and Nolan (2021) and Berto and Jago (2019) for more on hyperintensionality, and see Sedlár (2019) for a general framework for hyperintensional logics.

permit substitution into knowledge contexts. As a consequence, agents can know certain equivalences but this need not imply that their knowledge will be closed under substitution using those known equivalences. The logics are *congruential*, meaning they are closed under the rule (Cong), $A \leftrightarrow B \Rightarrow C(A) \leftrightarrow C(B)$. They are, therefore, not hyperintensional in the preferred sense of Odintsov and Wansing (2021).

To summarize, there are some ways in which knowledge in relevant epistemic logics runs into issues of logical omniscience and there are some ways in which it avoids issues of logical omniscience. To be clear, knowledge and knowers in all the epistemic logics under consideration is highly idealized. That idealization will bring with it some amount of logical omniscience, but it seems less bad than in the classical setting, because logical implication and equivalence are harder to come by in the relevant logical setting than in the classical one. One can, of course, combine these epistemic logics with modal operators interpreted using other techniques, such as justifications or other modal operators.[26] These additional techniques for interpreting modal operators would provide further alternative responses to issues of logical omniscience.

Importantly, since equivalence classes are used in the frames for Eq and CEq, one can make more or less direct contact with the extant work on common knowledge in classical partition frames, once one makes the straightforward extension to multi-agent systems where each agent is represented by their own equivalence relation which is used to interpret a distinct knowledge operator. There has been recent, pioneering work on common knowledge in relevant epistemic logics by Punčochář et al. (2023), but that work does not use equivalence relations for its accessibility relation, so it arguably has a different conception of knowledge. Given the importance of common knowledge in formal epistemology and epistemic logic, further investigation of this topic seems promising.

To end, I will highlight a lingering open question.

Open Can the logic Eq be axiomatized in the vocabulary $\{\to, \neg, \wedge, \vee, \Box\}$? If so, what axioms need to be added? If not, how could one show that?

It appears that one can axiomatize the logic if one adds Boolean negation ($-$) to the language, with the verification clause

- $a \Vdash -B$ iff $a \nVdash B$,

[26] For relevant logics of justifications, see Savić and Studer (2019) and Standefer (2019; 2022a; 2023c). For other modal operators interpreted using neighborhoods, see, for example, Standefer (2019) or Ferenz and Tedder (2022).

but there are philosophical reasons not to do that. Among other things, the variable-sharing property is violated when Boolean negation is added. Further, the attractive features of the logic Eq fall away in the presence of Boolean negation, e.g. (B) becomes valid when it is formulated using $-$ rather than $\neg$, namely as $A \rightarrow \Box - \Box - A$ rather than $A \rightarrow \Box\neg\Box\neg A$. For these reasons, an axiomatization that does not rely on Boolean negation would be preferable.

References

Anderson, A. R. and Belnap, N. D. (1975). *Entailment: The Logic of Relevance and Necessity, Vol. I.* Princeton University Press. 6, 9, 16, 21

Beall, J., Brady, R., Dunn, J. M., Hazen, A. P., Mares, E., Meyer, R. K., Priest, G., Restall, G., Ripley, D., Slaney, J., and Sylvan, R. (2012). On the ternary relation and conditionality. *Journal of Philosophical Logic*, 41(3):595–612. 7

Berto, F. (2015). A modality called 'negation'. *Mind*, 124(495):761–793. 17

Berto, F. and Jago, M. (2019). *Impossible Worlds*. Oxford University Press. 26

Berto, F. and Nolan, D. (2021). Hyperintensionality. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, spring 2021 edition. 26

Berto, F. and Restall, G. (2019). Negation on the australian plan. *Journal of Philosophical Logic*, 48(6):1119–1144. 17

Bílková, M., Majer, O., and Peliš, M. (2016). Epistemic logics for sceptical agents. *Journal of Logic and Computation*, 26(6):1815–1841. 6

Bilková, M., Majer, O., Peliš, M., and Restall, G. (2010). Relevant agents. In Beklemishev, L., Goranko, V., and Shehtman, V., editors, *Advances in Modal Logic*, number 8, pages 22–38. College Publications. 6

Bimbó, K. (2007). Relevance logics. In Jacquette, D., editor, *Philosophy of Logic*, volume 5 of *Handbook of the Philosophy of Science*, pages 723–789. Elsevier. 6

Brady, R., editor (2003). *Relevant Logics and Their Rivals, Volume II, A continuation of the work of Richard Sylvan, Robert Meyer, Val Plumwood and Ross Brady*. Ashgate. 16

Brady, R. T. (1994). Rules in relevant logic - I: Semantic classification. *Journal of Philosophical Logic*, 23(2):111–137. 3

Brady, R. T. (2017). Some concerns regarding ternary-relation semantics and truth-theoretic semantics in general. *IfCoLog Journal of Logics and Their Applications*, 4(3):755–781. 7

De, M. and Omori, H. (2018). There is more to negation than modality. *Journal of Philosophical Logic*, 47(2):281–299. 17

Dunn, J. M. (1993). Star and perp: Two treatments of negation. *Philosophical Perspectives*, 7:331–357. 17

Dunn, J. M. (1995). Positive modal logic. *Studia Logica*, 55(2):301–317. 17

Dunn, J. M. and Restall, G. (2002). Relevance logic. In Gabbay, D. M. and Guenthner, F., editors, *Handbook of Philosophical Logic*, volume 6, pages 1–136. Kluwer, 2nd edition. 6

Fagin, R. and Halpern, J. Y. (1987). Belief, awareness, and limited reasoning. *Artificial Intelligence*, 34(1):39–76. 6

Fan, J. (2023). Axiomatizing Rumsfeld ignorance. *Journal of Philosophical Logic*, pages 1–19. Forthcoming. 5

Ferenz, N. (2023). First-order relevant reasoners in classical worlds. *Review of Symbolic Logic*, pages 1–26. Forthcoming. 6

Ferenz, N. and Tedder, A. (2022). Neighbourhood semantics for modal relevant logics. *Journal of Philosophical Logic*. 27

Ferguson, T. M. (2017). *Meaning and Proscription in Formal Logic: Variations on the Propositional Logic of William T. Parry*. Cham, Switzerland: Springer Verlag. 26

Fine, K. (2018). Ignorance of ignorance. *Synthese*, 195(9):4031–4045. 5

Fuhrmann, A. (1990). Models for relevant modal logics. *Studia Logica*, 49(4):501–514. 11

Hawke, P., Özgün, A., and Berto, F. (2020). The fundamental problem of logical omniscience. *Journal of Philosophical Logic*, 49(4):727–766. 26

Hintikka, J. (1962). *Knowledge and Belief: an Introduction to the Logic of the Two Notions*. Cornell University Press, Ithaca, NY. 4, 5

Humberstone, L. (1988). Some epistemic capacities. *Dialectica*, 42(3):183–200. 4

Humberstone, L. (2010). Smiley's distinction between rules of inference and rules of proof. In Smiley, T. J., Lear, J., and Oliver, A., editors, *The Force of Argument: Essays in Honor of Timothy Smiley*, pages 107–126. Routledge. 3

Humberstone, L. (2011). *The Connectives*. MIT Press. 3

Humberstone, L. (2016). *Philosophical Applications of Modal Logic*. College Publications, London. 2, 4, 5

Levesque, H. J. and Lakemeyer, G. (2001). *The Logic of Knowledge Bases*. MIT Press. 6

Logan, S. A. (2024). *Relevance Logic*. Elements in Philosophy and Logic. Cambridge Univeristy Press. 6

Mares, E. D. (1992). The semantic completeness of RK. *Reports on Mathematical Logic*, pages 3–10. 21

Mares, E. D. (1993). Classically complete modal relevant logics. *Mathematical Logic Quarterly*, 39(1):165–177. 21

Mares, E. D. (1994). Mostly Meyer modal models. *Logique et Analyse*, 146:119–128. 19, 20, 23

Mares, E. D. (2004). *Relevant Logic: A Philosophical Interpretation*. Cambridge University Press. 6

Mares, E. D. (2020). Relevance Logic. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2020 edition. 6

Mares, E. D., Seligman, J., and Restall, G. (2010). Situations, constrains and channels. In van Benthem, J. and ter Meulen, A., editors, *Handbook of Logic and Language*, pages 329–344. Elsevier. 7

Meyer, J. C. and van der Hoek, W. (1995). *Epistemic Logic for AI and Computer Science*. Cambridge University Press. 1

Meyer, R. K. and Mares, E. D. (1993). Semantics of entailment 0. In Schroeder-Heister, P. and Došen, K., editors, *Substructural Logics*, pages 239–258. Oxford Science Publications. 7

Odintsov, S. and Wansing, H. (2021). Routley star and hyperintensionality. *Journal of Philosophical Logic*, 50(1):33–56. 27

Ono, H. (1977). On some intuitionistic modal logics. *Publications of RIMS Kyoto University*, 13:687–722. 23

Punčochář, V., Sedlár, I., and Tedder, A. (2023). Relevant epistemic logic with public announcements and common knowledge. *Journal of Logic and Computation*, 33(2):436–461. 6, 11, 27

Read, S. (1988). *Relevant Logic: A Philosophical Examination of Inference*. Blackwell. 6

Rendsvig, R., Symons, J., and Wang, Y. (2023). Epistemic Logic. In Zalta, E. N. and Nodelman, U., editors, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Winter 2023 edition. 1

Restall, G. (1999). Negation in relevant logics (how I stopped worrying and learned to love the Routley star). In Gabbay, D. M. and Wansing, H., editors, *What is Negation?*, pages 53–76. Kluwer Academic Publishers. 17, 18

Restall, G. (2000). *An Introduction to Substructural Logics*. Routledge. 9, 21

Restall, G. (2005). Logics, situations and channels. *Journal of Cognitive Science*, 6:125–150. 7

Robles, G. and Méndez, J. M. (2011). A class of simpler logical matrices for the variable-sharing property. *Logic and Logical Philosophy*, 20(3):241–249. 21

Robles, G. and Méndez, J. M. (2012). A general characterization of the variable-sharing property by means of logical matrices. *Notre Dame Journal of Formal Logic*, 53(2):223–244. 21

Routley, R., Plumwood, V., Meyer, R. K., and Brady, R. T. (1982). *Relevant Logics and Their Rivals*, volume 1. Ridgeview. 10

Savíc, N. and Studer, T. (2019). Relevant justification logic. *The IfCoLog Journal of Logics and their Applications*, 6(2). http://collegepublications.co.uk/ifcolog/?00031. 27

Sedlár, I. (2015). Substructural epistemic logics. *Journal of Applied Non-Classical Logics*, 25(3):256–285. 6

Sedlár, I. (2016). Epistemic extensions of modal distributive substructural logics. *Journal of Logic and Computation*, 26(6):1787–1813. 6

Sedlár, I. (2019). Hyperintensional logics for everyone. *Synthese*, 198(2):933–956. 26

Sedlár, I. and Vigiani, P. (2023). Relevant reasoning and implicit beliefs. In Hansen, H. H., Scedrov, A., and de Queiroz, R. J. G. B., editors, *Logic, Language, Information, and Computation: 29th International Workshop, WoLLIC 2023, Halifax, NS, Canada, July 11?14, 2023, Proceedings*, pages 336–350. Springer Nature Switzerland. 6

Stalnaker, R. (2006). On logics of knowledge and belief. *Philosophical Studies*, 128(1):169–199. 4, 5

Standefer, S. (2019). Tracking reasons with extensions of relevant logics. *Logic Journal of the IGPL*, 27(4):543–569. 27

Standefer, S. (2020). Actual issues for relevant logics. *Ergo*, 7(8):241–276. 21

Standefer, S. (2022a). A substructural approach to explicit modal logic. *Journal of Logic, Language and Information*. Forthcoming. 27

Standefer, S. (2022b). What is a relevant connective? *Journal of Philosophical Logic*, 51(4):919–950. 10

Standefer, S. (2023a). Hyperintensionality in relevant logics. In Alechina, N., Herzig, A., and Liang, F., editors, *Logic, Rationality, and Interaction: 9th International Workshop, LORI 2023, Jinan, China, October 26?29, 2023, Proceedings*, pages 238–250. Springer Nature Switzerland. 26

Standefer, S. (2023b). Varieties of relevant S5. *Logic and Logical Philosophy*, 32(1):53–80. 6, 10, 12, 13, 14, 16, 21

Standefer, S. (2023c). Weak relevant justification logics. *Journal of Logic and Computation*, 33(7):1665–1683. 27

Standefer, S. (202x). Variable-sharing as relevance. In Tedder, A., Sedlár, I., and Standefer, S., editors, *New Directions in Relevant Logics*. Springer. Forthcoming. 9

Standefer, S., Shear, T., and French, R. (2023). Getting some (non-classical) closure with justification logic. *Asian Journal of Philosophy*, 2(2):1–25. 6

Tedder, A. (2021). Information flow in logics in the vicinity of BB. *Australasian Journal of Logic*, 18(1):1–24. 7

Tedder, A. (2023). Situations, propositions, and information states. In Bimbó, K., editor, *Relevance Logics and other Tools for Reasoning: Essays in Honor of J. Michael Dunn*, pages 410–426. College Publications. 7

van Ditmarsch, H., Halpern, J. Y., van der Hoek, W., and Kooi, B. (2015). An introduction to logics of knowledge and belief. In van Ditmarsch, H., Halpern, J. Y., van der Hoek, W., and Kooi, B., editors, *Handbook of Epistemic Logic*, pages 1–51. College Publications. 1

van Ditmarsch, H., van der Hoek, W., and Kooi, B. (2007). *Dynamic Epistemic Logic*. Springer, Dordrecht, Netherland. 2

Williamson, T. (2000). *Knowledge and its Limits*. Oxford University Press, New York. 5

Williamson, T. (2006). Indicative versus subjunctive conditionals, congruential versus non-hyperintensional contexts. *Philosophical Issues*, 16(1):310–333. 26

Yap, A. (2014). Idealization, epistemic logic, and epistemology. *Synthese*, 191(14):3351–3366. 4