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ORIGINAL RESEARCH

Constructivism and Relevance

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Abstract
While constructivism and relevance characterize distinct movements in nonclassical logic, they are intimately related in ways that have not been fully appreciated. In this paper, we draw out some of these connections by focusing on certain heterodox strong relevant logics that embody both constructivist and relevant features. The relations between these logics and systems such as intuitionistic logic are examined from several perspectives, including semantically using operational models and proof-theoretically using natural deduction. Along the way, we illustrate how our approach yields novel insights into intuitionistic logic, and we highlight directions for future research.

1 Introduction

While relevant logics and constructive logics have often been viewed as having very different applications and motivations, there are many points of convergence. Both families of logics are subclassical, can be (and have been) motivated by information-theoretic considerations, and—at least for certain systems in certain fragments—have elegant proof-theoretic presentations that relax one or more structural features of classical logic. The connections between these two families have to some extent been obscured by the conventional focus on broadly classical accounts of negation in the relevant logic literature. Omitting negation (or, alternatively, adopting a constructive account of negation), the relationship between these families comes into clearer focus: all of the standard relevant logics are subsystems of intuitionistic logic and exhibit many of the same elegant constructive features (e.g., the disjunction property).

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Although this already suffices to justify the *technical* interest of a closer examination of the relations between relevant and constructive logics, it is worth emphasizing a further philosophical point. Conventional treatments of negation in relevant logic are broadly classical in the sense that they satisfy double negation elimination and all four De Morgan laws. Indeed, Anderson and Belnap adopted further classical negation principles (e.g., the law of excluded middle) for their preferred logics like R and E. While many relevant logicians study systems with weaker negation principles than those of R, the core remains broadly classical.[1]

However, many logicians and philosophers have raised objections to aspects of the broadly classical account of negation in relevant logic: its semantics (using the Routley star) is not entirely transparent,[2] it breaks certain conservative extension results which intuitively ought to hold,[3] and it (notoriously) fails to deliver disjunctive syllogism.[4] In addition, the essentially relevant features of relevant logic, in particular the variable-sharing property, do not rely on this picture of negation and are compatible with constructive negation (with some minor caveats).[5] This suggests that it was perhaps a mistake to align the relevant program with a broadly classical account of negation, and that, instead, it is more naturally aligned with constructivism.[6]

In this article, we survey points of convergence between relevant and constructive logics. We do this by focusing on two heterodox relevant logics in particular, $\mathsf{RM0}$ and S.[7] We shall observe that these systems exhibit a number of properties that have

[1] There has been some study of negations that do not obey some of the features mentioned above in relevant logic. For example, the logic BM, studied by Fuhrmann (1990) and Standefer et al. (2025), among others, gives up double negation elimination. The incompatibility-based negations studied by Restall (1999), Tabakci (2022), and others can give up double negation elimination and one of the De Morgan principles.

[2] See Copeland (1979, 1980) for this criticism, although see Restall (1999) for a defense of the Routley star.

[3] The addition of the negation principles of R to $\mathsf{RM0}$ result in a non-conservative extension thereof (see Anderson and Belnap (1975, §§8.15, 29.3) and Weiss (2022, p. 437, n. 3)).

[4] See Burgess (1981), although see, for example, Read (1981, 1983) for a defense of the relevantist position. One way of putting the objection is that, although the account of negation is broadly classical, it still cannot deliver some important classical inferences (e.g., disjunctive syllogism) on pain of irrelevance (but see Øgaard (2021a, 2021b)). For this reason, the account of negation may seem ad hoc. Of course, disjunctive syllogism is not a uniquely classical inference, and many of the constructive relevant logics we examine below validate it while still satisfying the variable-sharing property as defined in Definition 4. We note that in many relevant logics, disjunctive syllogism is an admissible rule, for which see, for example, Meyer and Dunn (1969), Weiss (2020), and Kripke (2022). There are, however, logics in which disjunctive syllogism fails to be admissible, for which see Meyer et al. (1984) and Øgaard and Standefer (2025). For some of the pertinent history, consult Urquhart (2016).

[5] Compare with the situation concerning classical and intuitionistic linear logic; see Di Cosmo and Miller (2023).

[6] On this point, compare with Urquhart (1972, §5) and Urquhart (1989). We note that Robles and Méndez (2018) provide ternary relational models for relevant logics with constructive, or intuitionistic-type, negations. We do not investigate ternary relational models in this paper and will not comment further on this work. Finally, we remark that Onishi (2016) develops an account of negation via implication that diverges from the sort of constructive negations considered in this paper.

[7] As will become clear, by S we intend the *semilattice* logic of Urquhart (1972), and not the *syllogistic* logic of Martin and Meyer (1982).

been deemed to be desirable by both relevantists and constructivists, and that they merit further study from logicians of both camps.

The plan of the article is as follows. In Sect. 2, we introduce the systems that we are interested in axiomatically, survey some definitions, and establish the elementary constructive and relevant credentials of several of the systems. We turn to operational-style semantics (model theory) in Sect. 3; a semilattice semantics for intuitionistic logic and S is examined in Sect. 3.1, and a bisemilattice semantics for intuitionistic logic, $\mathsf{RM0}$, and some of their neighbors is presented in Sect. 3.2. Proof-theoretic presentations of some of our logics using natural deduction are given in Sect. 4. Finally, we offer some concluding remarks in Sect. 5.

Before turning to logical matters proper, let us pin down some conventions regarding the language(s). By $\mathcal{L}(\mathcal{K})$, we mean the language built up from a countable set of atoms, $\Pi = \{p, q, \ldots\}$, possibly with subscripts, and the logical connectives in the set $\mathcal{K}$. We use $\mathcal{L}^+$ to stand for the basic positive language, that is, $\mathcal{L}(\{\to, \wedge, \vee\})$. $\mathcal{L}^\perp$ abbreviates $\mathcal{L}(\{\to, \wedge, \vee, \perp\})$, and similarly for $\mathcal{L}^{\mathsf{t}}$, $\mathcal{L}^{\mathsf{f}}$, etc. (t and f are the Ackermann truth and falsity constants). By $\mathcal{L}$ is just intended any of the languages extending $\mathcal{L}^+$, indifferently, or whichever is most contextually salient. As usual, $(\varphi \leftrightarrow \psi)$ is defined as $(\varphi \to \psi) \wedge (\psi \to \varphi)$. $\neg\varphi$ abbreviates $\varphi \to \perp$ and $-\varphi$ abbreviates $\varphi \to \mathsf{f}$. There will not be much occasion to examine $\top$, but it can be defined as $\perp \to \perp$ in any language containing $\perp$.

2 Logics

In this section we give a uniform presentation of the logics that we are interested in and assess some of their basic constructive and relevant credentials. Specifically, in Sect. 2.1, we give axiomatic presentations of our logics. We sketch proofs that many of these systems enjoy the disjunction property in Sect. 2.2 and that many of these systems enjoy the variable-sharing property in Sect. 2.3.

2.1 Axiom Systems

In this section, we present our logics via Hilbert-style axiom systems. The basic (positive) axioms and rules are as follows.

(Ax1) $\varphi \to \varphi$
(Ax2) $\varphi \to (\psi \to \varphi)$
(Ax3) $\varphi \to (\varphi \to \varphi)$
(Ax4) $(\varphi \to (\psi \to \gamma)) \to (\psi \to (\varphi \to \gamma))$
(Ax5) $(\varphi \to \psi) \to ((\psi \to \gamma) \to (\varphi \to \gamma))$
(Ax6) $(\varphi \to \psi) \to ((\gamma \to \varphi) \to (\gamma \to \psi))$
(Ax7) $(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$
(Ax8) $(\varphi \wedge \psi) \to \varphi$, $(\varphi \wedge \psi) \to \psi$
(Ax9) $\varphi \to (\varphi \vee \psi)$, $\psi \to (\varphi \vee \psi)$
(Ax10) $((\varphi \to \psi) \wedge (\varphi \to \gamma)) \to (\varphi \to (\psi \wedge \gamma))$
(Ax11) $((\varphi \to \gamma) \wedge (\psi \to \gamma)) \to ((\varphi \vee \psi) \to \gamma)$

(Ax12) $(\varphi \wedge (\psi \vee \gamma)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \gamma))$
(Ru1) $\varphi, \varphi \to \psi \Rightarrow \psi$
(Ru2) $\varphi, \psi \Rightarrow \varphi \wedge \psi$
(Ru3) $(\psi_0 \wedge ([\varphi_1 \wedge q_1, \ldots, \varphi_n \wedge q_n] \to \gamma)) \to ([\psi_1 \wedge q_1, \ldots, \psi_n \wedge q_n] \to \theta)$, $(\psi_0 \wedge ([\varphi_1 \wedge q_1, \ldots, \varphi_n \wedge q_n] \to \delta)) \to ([\psi_1 \wedge q_1, \ldots, \psi_n \wedge q_n] \to \theta)$ $\Rightarrow (\psi_0 \wedge ([\varphi_1, \ldots, \varphi_n] \to (\gamma \vee \delta))) \to ([\psi_1, \ldots, \psi_n] \to \theta)$

In the rules, the formulas to the left of the '$\Rightarrow$' are the premises and the formula to the right is the conclusion. $[\varphi_1, \ldots, \varphi_n] \to \varphi$ abbreviates $\varphi_1 \to (\varphi_2 \to \ldots (\varphi_n \to \varphi) \ldots)$. Rule (Ru3) is actually a rule schema ($n \geq 1$) and has the restriction that the q_i are all distinct and occur only as shown.[8] The basic systems that we are interested in are as follows.

- J^+ – all the axioms, (Ru1), and (Ru2).
- $\mathsf{RM0}^+$ – remove (Ax2) from J^+.
- S^+ – remove (Ax3) from $\mathsf{RM0}^+$, add (Ru3).
- R^+ – remove (Ax3) from $\mathsf{RM0}^+$.

Observe that the axiomatization of J^+ is particularly redundant, but it facilitates comparison (e.g., it makes clear that R^+ and $\mathsf{RM0}^+$ are sublogics of J^+).

Moving on, we will also be interested in extensions of these systems in languages including propositional constants of various kinds.

(Ax13) $\bot \to \varphi$
(Ax14) $\varphi \leftrightarrow (\mathsf{t} \to \varphi)$
(Ax15) $\varphi \vee (\varphi \to \mathsf{f})$

We adopt the following conventions hereafter. $\mathsf{L}^\bot$ is L^+ as formulated in $\mathcal{L}^\bot$ with the addition of axiom (Ax13), and similarly for L^t, L^f, etc. For example, J^+ is minimal logic (Johansson, 1937),[9] $\mathsf{J}^\bot$ is intuitionistic logic, and J^f is LD (Curry, 1950). By L is just intended any of the systems L^+, $\mathsf{L}^\bot$, etc., indifferently, or whichever is most contextually salient (by S we nearly always intend S^+; cf. footnote 33). By $\mathsf{L}(\mathcal{K})$ is intended the system in the language $\mathcal{L}(\mathcal{K})$ using the axioms for L pertaining to connectives in $\mathcal{K}$.

We define *proof* in the usual way: a proof of φ in the logic L is a finite sequence of formulas ending in φ such that each formula in the sequence is either an axiom or the conclusion of a rule of L the premises of which occur earlier in the sequence. If there is a proof of φ in L, we say that φ is a *theorem* of L (in symbols, $\vdash_\mathsf{L} \varphi$).

[8] Although we are not aware of any proof in the literature to the effect that S^+ is not finitely axiomatizable (i.e., axiomatizable using finitely many axiom schemata and rules), it does appear that a rule schema such as (Ru3) is essential. For axiomatizations of S^+, see Fine (1976), Charlwood (1978, 1981), and Kashima (2003).

[9] Sometimes minimal logic is presented with the distinguished constant $\bot$ but no axioms are postulated to govern it. It is clear that this is just a notational variant of J^+.

Before continuing, it is instructive to indicate how to actually *use* (Ru3) to derive an S^+ distinguishing theorem (i.e., a theorem distinguishing S^+ from R^+). Observe that both of the following are theorems of R^+:

1. $((\psi \to \gamma) \wedge ((\varphi \wedge q_1) \to \psi)) \to ((\varphi \wedge q_1) \to (\gamma \vee \delta))$;
2. $((\psi \to \gamma) \wedge ((\varphi \wedge q_1) \to \delta)) \to ((\varphi \wedge q_1) \to (\gamma \vee \delta))$.

The first follows from the transitivity for implication together with contraction (Ax7) and addition (Ax9), and the second from simplification (Ax8) and addition. Then for any q_1-free substitutions for the schematic variables, (Ru3) licenses the inference to:

3. $((\psi \to \gamma) \wedge (\varphi \to (\psi \vee \delta))) \to (\varphi \to (\gamma \vee \delta))$.

This is known not to be a theorem of R^+; see Urquhart (1972, p. 163).[10] However, it is not especially clear that this formula is objectionable on relevant grounds.

2.2 Disjunction Property

In this section, we sketch a proof that many of the previously defined systems enjoy the disjunction property, which is often taken as a hallmark of constructivism.[11]

Definition 1 A logic L has the *disjunction property* iff $\vdash_{\mathsf{L}} \varphi \vee \psi$ only if $\vdash_{\mathsf{L}} \varphi$ or $\vdash_{\mathsf{L}} \psi$.

As is well-known, $\mathsf{J}^{\perp}$ has the disjunction property. In fact, all of the constant-free logics described above (and many of them extended by constants apart from f) have the disjunction property.[12] While the f-containing logics cannot really be considered constructive, they nevertheless belong in the present study insofar as systems like J^{f} belong to a recognized family of broadly intuitionistic systems.

Our focus in this paper is on propositional logic, so the natural first-order analogue of the disjunction property, the existence property, is beyond the scope of the current study, although we briefly discuss it below. In Sect. 4, we turn to some related features of constructive logic, namely the Harrop property and Mints' rule.

The technique we employ below for establishing the disjunction property has a dual lineage in both constructive and relevant literature. This technique—*metavaluations*—combines algebraic valuations with provability in a proof system. Metavaluations were introduced in the context of modal logics and subsequently extended to non-modal logics by Meyer (1971, 1976). While Meyer's original metavaluations did not work with the De Morgan negation of standard relevant logics, alternative metavaluations were identified by Slaney (1984) that do work with De Morgan nega-

[10] However, it is admissible in R^+ in rule-form; that is, whenever $\vdash_{\mathsf{R}+} (\psi \to \gamma) \wedge (\varphi \to (\psi \vee \delta))$, it follows that $\vdash_{\mathsf{R}+} \varphi \to (\gamma \vee \delta)$. Thanks to Lloyd Humberstone for pointing this out.

[11] See Standefer (2022, §4), and references therein, for a discussion of the role of the disjunction property in constructive logic, and see pp. 929ff. for discussion of the variable-sharing property in relevant logics.

[12] The addition of De Morgan negation, in general, undermines the disjunction property. There are, however, many relevant logics weaker than R that have the disjunction property when De Morgan negation is added in the usual way (see, e.g., Slaney (1984)).

tion for many relevant logics. They have further been applied to modal relevant logics by Mares and Meyer (1992), Seki (2012, 2013), and Standefer and Mares (2025). Brady (2017a) provides an accessible overview of metavaluations.

Metavaluations are very similar to the technique of the *Kleene slash* (see, e.g., Kleene (1962)) that has been used in intuitionistic logic and mathematics. For example, Gabbay (1981, pp. 30ff.) uses a version of the Kleene slash to show first-order intuitionistic logic has various metatheoretical properties, and Božić and Došen (1984) use the Kleene slash in a study of intuitionistic modal logics. It appears that metavaluations and the Kleene slash were discovered independently.[13]

We begin by defining the canonical metavaluation for a logic L (cf. Meyer (1976, p. 504)).[14]

Definition 2 The *canonical metavaluation* v for a logic L is a function from the language to $\{0, 1\}$ obeying the following for all formulas φ, ψ.

- $v(p) = 0$, for atoms p;
- $v(\mathsf{t}) = 1$;
- $v(\bot) = 0$;
- $v(\varphi \wedge \psi) = 1$ iff $v(\varphi) = 1$ and $v(\psi) = 1$;
- $v(\varphi \vee \psi) = 1$ iff $v(\varphi) = 1$ or $v(\psi) = 1$;
- $v(\varphi \to \psi) = 1$ iff $\vdash_{\mathsf{L}} \varphi \to \psi$.

One can define metavaluations for theories by altering the atomic clause, although we will not do so here. One can also extend metavaluations to handle quantifiers, a point to which we will briefly return below. Using metavaluations, we define analogues of soundness and completeness.

Definition 3 Let v be the canonical metavaluation for the logic L.

L is *metasound* iff $\vdash_{\mathsf{L}} \varphi$ implies $v(\varphi) = 1$.
L is *metacomplete* iff $v(\varphi) = 1$ implies $\vdash_{\mathsf{L}} \varphi$.
L is *meta-adequate* iff it is both metasound and metacomplete.

Meyer (1976) showed that a wide range of logics are meta-adequate. Metacompleteness is straightforward:

Theorem 1 *Let* $\mathsf{L} \in \{\mathsf{J}^+, \mathsf{RM0}^+, \mathsf{R}^+, \mathsf{J}^\bot, \mathsf{RM0}^\bot, \mathsf{R}^\bot, \mathsf{J}^\mathsf{t}, \mathsf{RM0}^\mathsf{t}, \mathsf{R}^\mathsf{t}\}$ *and let* $\varphi \in \mathcal{L}$. *Then, if* $v(\varphi) = 1$, *then* $\vdash_{\mathsf{L}} \varphi$.

Proof The proof is by induction on the structure of φ. If φ is an atom or $\bot$, then $v(\varphi) = 0$, so the antecedent is always false. If φ is t, then the consequent holds because $\vdash_{\mathsf{L}} \mathsf{t}$ follows from (Ax1) and (Ax14). If φ is $\psi \wedge \theta$ or $\psi \vee \theta$, then the claim fol-

[13] Meyer (1976, pp. 514–515) notes several independent discoverers of the technique.

[14] Where t or $\bot$ are not in the language under consideration, the relevant clauses are omitted from the definition. As indicated above, f must be omitted.

lows easily from the inductive hypothesis. Finally, if φ is $\psi \to \theta$ and $v(\psi \to \theta) = 1$, then $\vdash_{\mathsf{L}} \psi \to \theta$ by Definition 2.

In order to prove metasoundness, Meyer introduced an auxiliary concept, the *canonical quasi-valuation* u for a logic L. This valuation is just like the canonical metavaluation except that the conditional clause is changed to the following.

- $u(\varphi \to \psi) = 1$ iff $\vdash_{\mathsf{L}} \varphi \to \psi$ and if $u(\varphi) = 1$, then $u(\psi) = 1$.

One can then prove the following lemma relating the canonical quasi-valuation and canonical metavaluation for a logic L.

Lemma 1 *Let v be the canonical metavaluation for L and u be the canonical quasi-valuation for L. Then for all formulas $\varphi \in \mathcal{L}$, if $u(\varphi) = 1$, then $v(\varphi) = 1$.*

Proof This is proved by induction on the complexity of φ. All the cases are immediate from the definition apart from the conditional case, which we now present. If $u(\psi \to \theta) = 1$, then $\vdash_{\mathsf{L}} \psi \to \theta$, which suffices for $v(\psi \to \theta) = 1$.

To prove metasoundness, we first prove soundness with respect to the canonical quasi-valuation (cf. Meyer (1976, pp. 508–509)).

Lemma 2 *Let $\mathsf{L} \in \{\mathsf{J}^+, \mathsf{RM0}^+, \mathsf{R}^+, \mathsf{J}^\perp, \mathsf{RM0}^\perp, \mathsf{R}^\perp, \mathsf{J}^\mathsf{t}, \mathsf{RM0}^\mathsf{t}, \mathsf{R}^\mathsf{t}\}$ and let u be the canonical quasi-valuation for L. Then for all formulas $\varphi \in \mathcal{L}$, if $\vdash_{\mathsf{L}} \varphi$, then $u(\varphi) = 1$.*

Proof The proof is by induction on the length of proof. We present two cases, (Ax2) and (Ru1). The initial provability clause (in the $\to$-condition for u) is satisfied for all conditional axioms where L contains the appropriate axiom.

Case (Ax2), that is, $\psi \to (\theta \to \psi)$: Suppose $u(\psi) = 1$ to show $u(\theta \to \psi) = 1$. By Theorem 1 and Lemma 1, $\vdash_{\mathsf{L}} \psi$. By (Ax2) and (Ru1), $\vdash_{\mathsf{L}} \theta \to \psi$. Since $u(\psi) = 1$ (ex hypothesi), if $u(\theta) = 1$, then $u(\psi) = 1$, whence $u(\theta \to \psi) = 1$ as desired.

Case (Ru1): Suppose the proof contains ψ and $\psi \to \theta$ and is extended by (Ru1). By the inductive hypothesis, $u(\psi) = 1$ and $u(\psi \to \theta) = 1$. It follows that if $u(\psi) = 1$ then $u(\theta) = 1$, so $u(\theta) = 1$, as desired.

Theorem 2 *Let $\mathsf{L} \in \{\mathsf{J}^+, \mathsf{RM0}^+, \mathsf{R}^+, \mathsf{J}^\perp, \mathsf{RM0}^\perp, \mathsf{R}^\perp, \mathsf{J}^\mathsf{t}, \mathsf{RM0}^\mathsf{t}, \mathsf{R}^\mathsf{t}\}$ and let $\varphi \in \mathcal{L}$. Then, if $\vdash_{\mathsf{L}} \varphi$, then $v(\varphi) = 1$.*

Proof Suppose $\vdash_{\mathsf{L}} \varphi$. By Lemma 2, $u(\varphi) = 1$, whence by Lemma 1, $v(\varphi) = 1$.

Putting Theorems 1 and 2 together, we get meta-adequacy for a range of logics.

Theorem 3 *Where* $\mathsf{L} \in \{\mathsf{J}^+, \mathsf{RM0}^+, \mathsf{R}^+, \mathsf{J}^\perp, \mathsf{RM0}^\perp, \mathsf{R}^\perp, \mathsf{J}^\mathsf{t}, \mathsf{RM0}^\mathsf{t}, \mathsf{R}^\mathsf{t}\}$, L *is meta-adequate.*

We note that it is not currently known whether S is meta-adequate. The sticking point is the (Ru3) case of Lemma 2.

Among the consequences of meta-adequacy is the disjunction property.

Theorem 4 *Suppose that* L *is meta-adeqauate. Then* L *enjoys the disjunction property.*

Proof Let v be the canonical metavaluation for L and suppose that $\vdash_\mathsf{L} \varphi \vee \psi$. Then $v(\varphi \vee \psi) = 1$, whence $v(\varphi) = 1$ or $v(\psi) = 1$ by Definition 2. This in turn implies $\vdash_\mathsf{L} \varphi$ or $\vdash_\mathsf{L} \psi$, as desired.

This is perhaps the easiest way of showing that relevant systems like $\mathsf{RM0}$ and R enjoy the disjunction property. While the technique can also be used to show that J enjoys the disjunction property, there are other methods for showing this, some of which are unavailable to the aforementioned relevant logics. S also enjoys the disjunction property, which we demonstrate using proof-theoretic methods in Sect. 4.[15]

While many logics are meta-adequate, it is worth noting that the positive fragment of classical logic is not. This can be seen by noting that $p \vee (p \to q)$ is a theorem of classical logic. The preceding theorem would then require that one of the disjuncts be a theorem, but neither is. Further, not every logic enjoying the variable-sharing property, discussed in Sect. 2.3, enjoys the disjunction property. In this way, it emerges that logics that are both constructive and relevant form a distinctive class.

Although our focus has been on propositional logics, we note that the technique of metavaluations extends straightforwardly to the first-order case. Meyer (1976) showed that first-order extensions of a variety of logics are meta-adequate. A consequence of this is that all such logics have the existence property.

Theorem 5 *If* L *is meta-adequate and* $\vdash_\mathsf{L} \exists x \varphi(x)$, *then there is some term t such that* $\vdash_\mathsf{L} \varphi(t/x)$.

Metavaluations have found some applications in the study of first-order relevant logics. Brady (2014) uses metavaluations to show that naïve set theory is consistent over certain relevant logics. Brady (2012) uses metavaluations to study a system of relevant arithmetic, establishing the existence property for this arithmetic. This result suggests that connections between relevant and constructive logics should be further studied in the context of theories of arithmetic.

[15] West and Weiss (2026) give a semantic proof of the disjunction property for S.

2.3 Variable-Sharing Property

Now we turn to the variable-sharing property, which is often taken as a necessary condition on being a relevant logic.[16]

Definition 4 A logic L has the *variable-sharing property* just in case $\vdash_{\mathsf{L}} \varphi \to \psi$ only if φ and ψ share an atom, where φ and ψ do not contain any propositional constants.[17]

Let us say that a logic L is a *theorem-sublogic* of a logic M iff $\{\varphi \in \mathcal{L} :\vdash_{\mathsf{L}} \varphi\} \subseteq \{\varphi \in \mathcal{L} :\vdash_{\mathsf{M}} \varphi\}$. It is clear that the variable-sharing property is preserved to logics that are theorem-sublogics of a logic with the variable-sharing property.

We now briefly show that $\mathsf{RM0}$ enjoys the variable-sharing property. For this, we will use matrix methods. A matrix is a triple $\langle V, D, O\rangle$, where V is a set of values, $D \subseteq V$ is a non-empty set of designated values, and O is a set of operations on V. A valuation v on a matrix maps atoms to V and is extended to a homomorphism on the whole language via the operations in O. Given a matrix M, a valuation v on M is a counterexample to a formula φ iff $v(\varphi) \notin D$. Finally, a formula φ is valid on a matrix if no valuation on the matrix is a counter example to φ.

$\to$	0	.5	1
0	1	1	1
.5	0	.5	1
1	0	0	1

$\wedge$	0	.5	1
0	0	0	0
.5	0	.5	.5
1	0	.5	1

$\vee$	0	.5	1
0	0	.5	1
.5	.5	.5	1
1	1	1	1

The matrix *RM*3 uses the tables above to interpret the connectives and $D = \{.5, 1\}$ (the constants, if included, can be interpreted as follows: $v(\mathsf{t}) = .5$ and $v(\mathsf{f}) = v(\bot) = 0$). It follows from a result of Meyer that all the axioms of $\mathsf{RM0}$ are valid on *RM*3 and all its rules preserve validity.[18]

Theorem 6 $\mathsf{RM0}$ *enjoys the variable-sharing property.*[19]

Proof Suppose φ and ψ share no atoms and contain no occurrences of propositional constants. Then consider any valuation v on *RM*3 such that $v(p) = 1$ for all p in φ, and $v(q) = .5$ for all q in ψ. One can show by induction on the structure of formulas that $v(\varphi) = 1$ and $v(\psi) = .5$. Since $1 \to .5 = 0$, it follows that $\varphi \to \psi$ is not a theorem of $\mathsf{RM0}$.

[16] Anderson and Belnap (1975, p. 33) say that variable-sharing is necessary but not sufficient for the relevance of φ to ψ. Standefer (2025) proposes taking the variable-sharing property to be sufficient as well.

[17] Equivalently, where the system in question conservatively extends L^+, just in case $\vdash_{\mathsf{L}^+} \varphi \to \psi$ only if φ and ψ share an atom.

[18] See Anderson and Belnap (1975, p. 470).

[19] This result appears to have first been observed in the literature by Méndez (1988, p. 286). Note that the matrices he uses are notational variants of those given here.

This theorem shows that RM0 has the variable-sharing property,[20] and so, therefore, do R and all its sublogics as well.[21] S also enjoys the variable-sharing property. Since it is not known whether S is a theorem-sublogic of RM0, Theorem 6 cannot be used to show that S enjoys the variable-sharing property. An alternative proof was provided by Weiss (2019). J does not have the variable-sharing property, since it has $p \to (q \to q)$, for example, as a theorem.

It is worth noting, for contrast, that the disjunction property is not automatically preserved to theorem-sublogics, since the provability claims in the consequent may fail even though the antecedent is satisfied.[22] To illustrate the idea, consider the logic whose only rule is (Ru1) and whose only axiom is $(\varphi \to \varphi) \vee (\psi \to \psi)$. This logic is a theorem-sublogic of J, but it lacks the disjunction property, as it has no non-disjunctive theorems (so, in particular, $p \to p$ is not a theorem).

3 Operational Semantics

In this section, we survey two kinds of operational model theory with an affinity to the informal BHK (Brouwer-Heyting-Kolmogorov) semantics. Both can be used to model not only J but also certain heterodox relevant logics like S and RM0. To motivate these frameworks, let us briefly review the BHK semantics. The semantics for the propositional language of intuitionistic logic is as follows[23]:

(i) A proof of $\varphi \wedge \psi$ is a combination of a proof of φ and a proof of ψ;
(ii) A proof of $\varphi \vee \psi$ is given by giving either a proof of φ or a proof of ψ;
(iii) A proof of $\varphi \to \psi$ is a construction which, given a proof of φ, returns a proof of ψ;
(iv) There is no proof of $\bot$.

Although this is only an informal semantics, it already suffices to motivate the rejection of characteristically classical theses like the law of the excluded middle, $\varphi \vee \neg\varphi$. For if there is no proof of φ and no proof of $\neg\varphi$ (i.e., $\varphi \to \bot$), then of course there is no proof of their disjunction either.

The semantic frameworks that follow will, to varying degrees, *internalize* this informal semantics (cf. Fine (2014)). The connections will be particularly sharp for

[20] In fact, the theorem shows something stronger, namely that the set of formulas valid on the *RM*3 matrix enjoys the variable-sharing property. Among the valid formulas is $\varphi \vee (\varphi \to \psi)$. It is clear that the formula $p \vee (p \to q)$ is valid, while neither disjunct is. Consequently, the logic of this matrix is not meta-adequate (Definition 3), even though it enjoys the variable-sharing property.

[21] In fact, R has the variable-sharing property even when De Morgan negation is included in the language with its standard axioms. RM0 does not have the variable-sharing property if the negation axioms of R are added; this results in the properly stronger system RM (for further discussion, see footnote 42). However, Méndez (1988, pp. 290–293) gives an extension of RM0 with a sort of weak De Morgan negation that retains the variable-sharing property.

[22] Compare with the so-called weak variable-sharing property, for which see Øgaard (2023).

[23] There is some variation in how these conditions are presented; consult, for example, Heyting (1956, §7.1.1), van Dalen (1986, p. 231), Troelstra and van Dalen (1988a, p. 9), Artemov (2001, pp. 1–2), Fine (2014, p. 550), and Weiss (2021b, p. 173).

the semilattice semantics articulated in Sect. 3.1. The bisemilattice semantics of Sect. 3.2 will then emerge as a natural generalization thereof. Both frameworks retain a form of what is arguably most distinctive of the BHK semantics, namely, its clause for the conditional.

3.1 The Semilattice Semantics

In this section, we present a formal model theory that was initially proposed and extensively developed by Urquhart (1971, 1972, 1973).[24] Drawing inspiration from Anderson and Belnap's subscripted, or indexed, natural deduction systems (Anderson, 1960; Anderson and Belnap, 1962), Urquhart developed the semilattice semantics and variations on it to model (fragments of) relevant logics like R, though he also pointed out how a slight modification of the semantics for R can be used to model (a fragment of) J.

Definition 5 A *semilattice frame* is a structure $\mathfrak{F} = \langle S, 0, \sqcup \rangle$ where $\langle S, \sqcup \rangle$ is a join-semilattice and $0 \in S$ is lattice bottom.

Urquhart (1972, pp. 159–160) interpreted the constituents of a semilattice frame informationally: the elements of S are thought of as pieces of information, 0 is the null piece of information, and $\sqcup$ is an operation for pooling information. However, the BHK semantics suggests a distinct (and, ultimately, we believe, superior) interpretation of this framework according to which the elements of S are understood to be proofs (or constructions), 0 is the null proof, and $\sqcup$ is an operation for proof combination.[25]

We first present the models for J, and later examine how to obtain a relevant neighbor thereof.

Definition 6 A J *semilattice model* is a structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ where $\mathfrak{F} = \langle S, 0, \sqcup \rangle$ is a frame and $V : \Pi \cup \{\bot\} \to \mathcal{P}(S)$ is subject to the conditions that:

1. $x \in V(\mathfrak{a})$ implies $x \sqcup y \in V(\mathfrak{a})$, $\mathfrak{a} \in \Pi \cup \{\bot\}$;
2. $x \in V(\bot)$ implies $x \in V(p)$ (for all p).

Where $\mathfrak{M} = \langle S, 0, \sqcup, V \rangle$ is a J semilattice model and $x \in S$, the relation $\mathfrak{M} \models_x$ is defined as follows:

$\mathfrak{a}$. $\mathfrak{M} \models_x \mathfrak{a}$ if and only if $x \in V(\mathfrak{a})$, for $\mathfrak{a} \in \Pi \cup \{\bot\}$;
$\wedge$. $\mathfrak{M} \models_x \varphi \wedge \psi$ if and only if $\mathfrak{M} \models_x \varphi$ and $\mathfrak{M} \models_x \psi$;
$\vee$. $\mathfrak{M} \models_x \varphi \vee \psi$ if and only if $\mathfrak{M} \models_x \varphi$ or $\mathfrak{M} \models_x \psi$;

[24] For an overview of this semantics and variations on it, consult Standefer (2021a). The presentation given here largely follows Weiss (2021b).

[25] The sort of combination we envision is that exemplified by combining proofs of lemmata (or sub-claims) into proofs of theorems. Such a form of combination intuitively satisfies all the properties of a join operation (associativity, commutativity, and idempotence).

$\to$. $\mathfrak{M} \models_x \varphi \to \psi$ if and only if for all $y \in S$, $\mathfrak{M} \not\models_y \varphi$ or $\mathfrak{M} \models_{x \sqcup y} \psi$.

In the BHK spirit, $\mathfrak{M} \models_x \varphi$ is to be read as: x is a proof of (i.e., construction establishing) φ (in $\mathfrak{M}$). That this relation is *inexact*, in the sense of Fine (2017, p. 558), is indicated by the following result, which has a routine proof by induction.

Lemma 3 *For any* J *semilattice model* $\mathfrak{M} = \langle S, 0, \sqcup, V\rangle$*, any formula* φ*, and any* $x, y \in S$*,* $\mathfrak{M} \models_x \varphi$ *implies* $\mathfrak{M} \models_{x\sqcup y} \varphi$*.*

The connections between these truth conditions and the aforementioned BHK conditions should be more-or-less transparent, though the cases of $\bot$ and $\wedge$ call for brief comment. The standard truth condition for $\bot$ from Kripke (1965)—$\mathfrak{M} \not\models_x \bot$ always—over-generates over the semilattice semantics (basically, because the partial order induced by $\sqcup$ is directed). In particular, the weak law of excluded middle, $\neg\varphi \vee \neg\neg\varphi$, comes out valid, so the resulting logic is (at least as strong as) the intermediate logic KC.[26] The truth condition adopted in its place (cf. Veldman (1976), Fine (2014, p. 565)), in conjunction with the conditions imposed in Definition 6, more nearly matches the notion that a proof of $\bot$ is a proof of everything (cf. Ax13).[27]

The BHK truth condition for $\wedge$ really has an intensional character (a *combination* of proofs) whereas the truth condition given above is plainly extensional. In view of Lemma 3, the following intensional condition is equivalent to it:

$\wedge'$. $\mathfrak{M} \models_x \varphi \wedge \psi$ if and only if $\exists y, z \in S$ such that $x = y \sqcup z$, $\mathfrak{M} \models_y \varphi$ and $\mathfrak{M} \models_z \psi$.

Indeed, this is the truth condition that Urquhart (1972, p. 164) assigns to intensional conjunction, or fusion (cf. van Fraassen (1969, p. 484)).

It is natural to define *validity* in terms of what is verified by the null proof, 0.[28] More precisely, we write $\mathfrak{M} \models \varphi$ if $\mathfrak{M} \models_0 \varphi$; $\mathfrak{F} \models \varphi$ if $\mathfrak{M} \models \varphi$ for all J semilattice models $\mathfrak{M}$ over the frame $\mathfrak{F}$; and $\models_{\mathsf{J}} \varphi$ if $\mathfrak{F} \models \varphi$ for all semilattice frames $\mathfrak{F}$.

The following results are proved in Weiss (2021a, 2021b), or are easy corollaries or refinements of results proved therein.

Theorem 7 $\vdash_{\mathsf{J}} \varphi$ *if and only if* $\models_{\mathsf{J}} \varphi$*.*[29]

[26] For KC (sometimes known as Jankov's logic), see Dummett and Lemmon (1959) and Jankov (1968a). For the operational completeness of KC, see Weiss (2021b, §4).

[27] It is easily proved that for any formula φ, $x \in V(\bot)$ implies $\mathfrak{M} \models_x \varphi$ (Weiss, 2021b, p. 177). This semantic fact accords with the definition of $\bot$ as $(\forall p)p$ sometimes found in the context of propositionally quantified relevant logic (cf. Anderson and Belnap (1961, p. 721) and Meyer (1973, p. 187)).

[28] In view of Lemma 3, the more usual definition in terms of what is verified by all proofs $x \in S$ is equivalent, however.

[29] Note that certain fragments of $\mathsf{RM0}$—for example, $\mathsf{RM0}(\to, \wedge)$—can also be characterized using semilattice frames and a weak form of heredity ($x \in V(p)$ and $y \in V(p)$ imply $x \sqcup y \in V(p)$); see Urquhart (1972, p. 168). There is no known way of characterizing even $\mathsf{RM0}^+$ in this framework, however.

Theorem 8 $\vdash_{\mathsf{J}} \varphi$ *if and only if* $\mathfrak{F} \models \varphi$ *for all finite semilattice frames* $\mathfrak{F}$.[30]

Theorem 9 $\vdash_{\mathsf{J}} \varphi$ *if and only if* $\mathfrak{F} \models \varphi$ *for all finitely atomistic semilattice frames* $\mathfrak{F}$.[31]

Theorem 10 $\vdash_{\mathsf{J}} \varphi$ *if and only if* $\langle \mathbb{N}, 1, lcm \rangle \models \varphi$*, where* $\langle \mathbb{N}, 1, lcm \rangle$ *is the divisibility lattice.*

Theorem 7 is the basic semilattice soundness and completeness result for J. Theorem 8 gives the finite model property for J with respect to its semilattice semantics. It should be observed that this result is nontrivial to the extent that, while J has the finite model property with respect to its Kripke semantics (Troelstra and van Dalen, 1988a, p. 92) and with respect to its Heyting algebraic semantics (Troelstra and van Dalen, 1988b, p. 708), it does *not* with respect to (certain forms of) the Beth semantics (Humberstone, 2011, §6.43). Theorem 9 has a nice constructivist philosophical gloss: J is characterized by those spaces of proofs (or constructions) in which every proof is itself a finite agglomeration of certain basic proofs. Theorem 10 realizes for J a sort of arithmetical interpretation of logical reasoning that Leibniz famously sought for the syllogistic (see Couturat (1903, pp. 42ff.), and for some pertinent discussion, Weiss (2026)).

While the semilattice semantics can be used to characterize non-relevant systems such as J, it was conceived for modeling relevant systems such as R and E. The most straightforward way of "relevantizing" the semantics, and obtaining a constructive relevant logic on its basis, is to drop both of the special constraints from Definition 6 (so, a *semilattice model* is a pair consisting of a frame and any interpretation whatsoever).[32] This strategy, already suggested by Urquhart (1972, pp. 165–166), effectively treats $\bot$ (and, derivatively, $\neg$) as in minimal logic (Johansson, 1937). For convenience, we here focus on S as formulated in $\mathcal{L}^+$ (i.e., S^+), distinguishing an atom to serve the role of a minimal $\bot$ if so desired.[33]

Let us write $\models_{\mathsf{S}} \varphi$ if φ is true at 0 in all semilattice models (as defined in the previous paragraph). An axiomatization of S was sketched by Fine (1976) and a detailed completeness proof for S with respect to $\models_{\mathsf{S}}$ was given by Charlwood (1978, 1981) (the axiomatization given in Sect. 2.1 is essentially the same as Charlwood's). Although $\mathsf{S} \supset \mathsf{R}$ (Urquhart, 1972, p. 163), these systems do coincide in certain fragments; for example, $\mathsf{S}(\rightarrow, \wedge) = \mathsf{R}(\rightarrow, \wedge)$.

Semantically speaking, S is an extremely elegant logic, and it turns out to be quite elegant from a certain proof-theoretic perspective as well (see Sect. 4). However, it is

[30] Proof sketch: By the finite model property for J with respect to its Kripke semantics, if $\nvdash_{\mathsf{J}} \varphi$, then φ fails in a finite Kripke model. Use the unravelling construction from Kripke (1965, p. 110) (Theorem 1, second part) to show that φ fails in a finite rooted tree model. From here, the argument proceeds as in Weiss (2021b, pp. 182–184).

[31] An *atom* in a semilattice with a least element 0 is an element x such that $0 \sqsubset x$ and for any y such that $0 \sqsubseteq y \sqsubseteq x$, $y = 0$ or $y = x$. A semilattice frame is *finitely atomistic* if every element is a finite (possibly null) join of atoms.

[32] Some alternative ways of obtaining constructive relevant logics are examined by Weiss (2021b, §5).

[33] There is no canonical account of negation for S so this system is usually presented in a purely positive language.

not particularly elegant from an axiomatic perspective, nor does it coincide (except in certain fragments) with any of the systems that relevant logicians were pre-semantically interested in.

It is an interesting philosophical question as to whether this really matters. Axiom chopping does not seem to be a particularly reliable guide to capturing the canons of good reasoning (a point implicitly or explicitly accepted both by model-theoretic semanticists, who emphasize models, and proof-theoretic semanticists, who generally emphasize sequent or natural deduction systems). Just as many of the Lewis modal systems, originally presented axiomatically, have fallen by the wayside in favor of other modal systems with more elegant semantic presentations, perhaps the relevant logician ought to embrace semantically elegant systems like S and give up systems like R, which are older and axiomatically elegant, but semantically (and proof-theoretically) rather inelegant.[34]

We offer the foregoing as food for thought, but will not pursue the line any further here. In Sect. 3.2, we instead pursue a more moderate path by elaborating a semantics that complicates that of S somewhat while also yielding a slightly more (axiomatically) conventional logic. We leave to the reader to judge the relative costs and merits of the two approaches.

3.2 The Bisemilattice Semantics

In this section we examine a semantics from Weiss (2022), which is in turn a variation on a semantics due to Humberstone (1988) (cf. Fine (1974), Ono and Komori (1985), and Došen (1988, 1989)). This semantics characterizes $\mathsf{RM0}$, and a natural special case of it characterizes J. We discuss both below.

By way of motivating this framework, it will be instructive to scrutinize the truth condition for disjunction from the semilattice semantics (Sect. 3.1) in connection with different informal interpretations of that semantic machinery a bit more closely. The condition—that $\mathfrak{M} \models_x \varphi \vee \psi$ if and only if either $\mathfrak{M} \models_x \varphi$ or $\mathfrak{M} \models_x \psi$—makes good sense on the BHK-style interpretation articulated above coupled with a constructivist notion of proof. But consider again the informational interpretation of the semantics advanced by Urquhart (1972). On this interpretation, $\mathfrak{M} \models_x \varphi$ means that the information x supports, or determines, that φ; accordingly, for a piece of information x to support $\varphi \vee \psi$ it must be the case that x either supports φ or supports ψ. However, it is a seemingly mundane fact that a piece of information, or a theory, can support a disjunction without thereby supporting, or determining, either disjunct (cf. Copeland (1979, p. 408), Humberstone (1988, pp. 64–65), and Humberstone (2011, §6.45)). After all, a witness to a crime can testify that the butler did it or the chauffeur did it—that's information that supports a disjunction—without thereby providing testimony that of itself determines which.

[34] For criticisms of various parts of the ternary relational semantics and proof theory of R (and other standard relevant logics), see Urquhart (1972, 1989), Copeland (1979, 1980), and Brady (2017b). Incidentally, a number of these criticisms center on the artificiality of the treatment of non-constructive negation in the semantics (cf. Sect. 1), and so are clearly inapplicable to S.

If one is attracted to the informational interpretive scheme, this suggests that one should countenance another form of combination to interpret disjunction. The idea, roughly, is that for a piece of information to support a disjunction is for it to pare down or fuse pieces of information supporting the disjuncts into something that *merely* suffices to determine that one or the other disjunct holds. The rest of the positive language is to be interpreted, roughly, in accordance with semilattice semantics, although some further technical conditions are required to ensure that the distinct pieces of the machinery interact correctly. We turn now to the formal details.

The core of the semantics is an elaboration of the notion of a frame from the semilattice semantics. Whereas in that semantics frames are semilattices, in this semantics frames are *bisemilattices*. Roughly, a bisemilattice is just two cohabitating semilattices. Caveat lector: Every lattice is a bisemilattice, but not conversely. In a lattice—though not in a bisemilattice—the operations of the two constituent semilattices must interact in the manner specified by the absorption laws.[35]

Nevertheless, in what follows, we will be interested in bisemilattices in which the operations' interactions are constrained in certain ways.

Definition 7 A RM0 *bisemilattice frame* is a structure $\mathfrak{F} = \langle S, 0, 1, \sqcup, \sqcap \rangle$ where:

1. Both $\langle S, 0, \sqcup \rangle$ and $\langle S, 1, \sqcap \rangle$ are bounded semilattices;
2. $x \sqcap y \sqsubseteq_\sqcap z \Rightarrow \exists x', y' : x \sqsubseteq_\sqcap x', y \sqsubseteq_\sqcap y'$, and $z = x' \sqcap y'$;
3. $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$;
4. $x \sqsubseteq_\sqcup 1$.

We offer a few additional remarks on Definition 7 before continuing.[36] The two orders, $\sqsubseteq_\sqcap$ and $\sqsubseteq_\sqcup$, are in general distinct in a bisemilattice (this cannot happen in a lattice, in which $\sqsubseteq$ is unambiguous). These relations are defined in the usual way ($x \sqsubseteq_\sqcap y$ iff $x \sqcap y = x$; $x \sqsubseteq_\sqcup y$ iff $x \sqcup y = y$). Consequently, what boundedness comes to in condition 1 is that $0 \sqsubseteq_\sqcup x$ and $x \sqsubseteq_\sqcap 1$ for all $x \in S$. *Top respect* (condition 4) is indeed something more than boundedness. Finally, note that *meet-decomposability* (condition 2) and *join-distributivity* (condition 3) are equivalent over lattices, but not over bisemilattices.

How is this machinery to be interpreted? We have already indicated how it might be understood informationally, but another possible interpretation is suggested by Fine (1974) (cf. Logan (2024)).[37] Given a frame $\mathfrak{F} = \langle S, 0, 1, \sqcup, \sqcap \rangle$, we think of the elements of S as theories, 0 and 1 as special distinguished theories (the base and trivial theory, respectively), and $\sqcup$ and $\sqcap$ as certain ways of "actively" and "passively" combining theories, respectively. Roughly, the idea is that we can think of $x, y \in S$ as collections of inert propositions, and combine them passively to get the collection of propositions *in* both ($x \sqcap y$); or we can think of them as collections of actionable propositions, and combine them actively to get the theory *generated* by both ($x \sqcup y$).

[35] That is, $x \sqcup (x \sqcap y) = x$ and $x \sqcap (x \sqcup y) = x$.

[36] For further details, consult Weiss (2022, §3.1).

[37] This interpretation is also suggested by the canonical model construction from Weiss (2022, §3.3).

On the foregoing interpretation, the conditions that are imposed in Definition 7 are fairly transparent, but also the conditions that are not imposed.[38] For example, it is not imposed that, for all $x \in S$, $0 \sqsubseteq_{\sqcap} x$. Thinking of 0 as the theory of the logic itself (as, indeed, it is in the canonical model construction from Weiss (2022, §3.3)), what this condition would come to, intuitively, is that every theory (arithmetic, biology, etc.) is *about* logic. This is not a very relevant line of thought. For the relevantist, a theory can be logically well-structured without being about logic—without even commenting on logic at all.[39]

We have just been suggesting that $0 \sqsubseteq_{\sqcap} x$ ought not to be required. If, however, it is imposed on a bisemilattice satisfying the conditions in Definition 7, the bisemilattice must be a lattice—indeed, a bounded distributive lattice. Such structures, in turn, furnish a semantic framework for **J** (cf. Došen (1989, §3.8)).

Definition 8 A **J** *bisemilattice frame* is a structure $\mathfrak{F} = \langle S, 0, 1, \sqcup, \sqcap \rangle$ where this is just a bounded distributive lattice.

Observe that, in this framework, **J** and **RM0** are distinguished by their frames, whereas in the semilattice semantics, **J** and **S** are distinguished by the conditions imposed on their models.

Before turning to models, we recall that a *filter* in a bisemilattice $\langle S, \sqcup, \sqcap \rangle$ is a set $\emptyset \neq F \subseteq S$ such that $x, y \in F$ if and only if $x \sqcap y \in F$. We write $\mathcal{F}(\mathfrak{F})$ for the set of all filters in the bisemilattice $\mathfrak{F}$.

Definition 9 A *bisemilattice model* is a structure $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ where $\mathfrak{F}$ is a bisemilattice frame and $V : \Pi \to \mathcal{F}(\mathfrak{F})$.

Naturally, a **J** (**RM0**) model is just a bisemilattice model based on a **J** (**RM0**) bisemilattice frame. Where $\mathfrak{M} = \langle S, 0, 1, \sqcup, \sqcap, V \rangle$ is a bisemilattice model and $x \in S$, the relation $\mathfrak{M} \models_x$ is defined as follows.

[40]p. $\mathfrak{M} \models_x p$ if and only if $x \in V(p)$;
$\bot$. $\mathfrak{M} \models_x \bot$ if and only if $x = 1$;
t. $\mathfrak{M} \models_x$ t if and only if $0 \sqsubseteq_{\sqcap} x$;
f. $\mathfrak{M} \models_x$ f if and only if $x \not\sqsubseteq_{\sqcap} 0$;
$\wedge$. $\mathfrak{M} \models_x \varphi \wedge \psi$ if and only if $\mathfrak{M} \models_x \varphi$ and $\mathfrak{M} \models_x \psi$;
$\vee$. $\mathfrak{M} \models_x \varphi \vee \psi$ if and only if $\exists y, z \in S$ such that $\mathfrak{M} \models_y \varphi$, $\mathfrak{M} \models_z \psi$, and $x = y \sqcap z$;

[38] Roughly similar points can be made using the informational interpretation (0 is the information conveyed by logic itself, the other elements of S represent the information conveyed by other theories, etc.), but we find this somewhat less natural.

[39] Hence, in relevant logic, all theories are logically well-behaved in certain respects (e.g., closure under adjunction), but not all theories are *regular* (i.e., contain all logical truths). See Mares (2024) for more on this idea.

[40] It is worth pointing out that, in any **J** bisemilattice model, the truth condition for f simplifies as follows (because 0 is the least element in $\sqsubseteq_{\sqcap}$):$\mathfrak{M} \models_x$ f if and only if $x \neq 0$. This is essentially what the corresponding truth condition given in Segerberg (1968, pp. 37–38) comes to (0 being the sole "normal" world).

$\rightarrow$. $\mathfrak{M} \models_x \varphi \rightarrow \psi$ if and only if for all $y \in S$, $\mathfrak{M} \not\models_y \varphi$ or $\mathfrak{M} \models_{x \sqcup y} \psi$.

We write $\mathfrak{M} \models \varphi$ if $\mathfrak{M} \models_0 \varphi$; $\mathfrak{F} \models \varphi$ if $\mathfrak{M} \models \varphi$ for all models $\mathfrak{M}$ over the frame $\mathfrak{F}$; and $\models_{\mathsf{J}} \varphi$ ($\models_{\mathsf{RM0}} \varphi$) if $\mathfrak{F} \models \varphi$ for all J ($\mathsf{RM0}$) frames $\mathfrak{F}$. The following results are proved for the f-free systems by Weiss (2022, §3.3).

Theorem 11 $\vdash_{\mathsf{J}} \varphi$ *if and only if* $\models_{\mathsf{J}} \varphi$.

Theorem 12 $\vdash_{\mathsf{RM0}} \varphi$ *if and only if* $\models_{\mathsf{RM0}} \varphi$.

These results can be extended to systems containing f but the required modifications are not entirely straightforward. We sketch the amendments here for the case of $\mathsf{RM0}^{\mathsf{f}}$, but we omit the details. Importantly, it is no longer possible to construct a single countermodel to all non-theorems. Instead, for a given non-theorem φ, it will be shown that an $\mathsf{RM0}^{\mathsf{f}}$-theory can be constructed which excludes φ but also satisfies certain further properties.

Let us call Γ a *normal* $\mathsf{RM0}^{\mathsf{f}}$*-theory* if it is a theory (i.e., Γ is closed under adjunction and provable conditionals), it is prime ($\varphi \vee \psi \in \Gamma \Rightarrow \varphi \in \Gamma$ or $\psi \in \Gamma$), it is regular ($\mathsf{RM0}^{\mathsf{f}} \subseteq \Gamma$), and it is consistent ($\mathsf{f} \notin \Gamma$). The proof of the following lemma is involved, but uses routine techniques (see, e.g., Mares (1995, §5)):

Lemma 4 *If* $\not\vdash_{\mathsf{RM0}^{\mathsf{f}}} \varphi$*, then there is a normal* $\mathsf{RM0}^{\mathsf{f}}$*-theory* Γ *such that* $\varphi \notin \Gamma$.

For the chosen non-theorem φ, use Lemma 4 to cook up a normal $\mathsf{RM0}^{\mathsf{f}}$-theory Γ excluding φ. A Γ-*theory* is a set Δ closed under adjunction and conditionals in Γ (i.e., $\psi \in \Delta$ and $\psi \rightarrow \theta \in \Gamma \Rightarrow \theta \in \Delta$). The countermodel to φ is now defined as in Weiss (2022, §3.3) but using the notion of a Γ-theory in place of an L-theory.[41]

Definition 10 For a given normal $\mathsf{RM0}^{\mathsf{f}}$-theory Γ, define $\mathfrak{M}^\Gamma = \langle \mathbb{TH}, \Gamma, \mathcal{L}^{\mathsf{f}}, \cdot, \cap, V \rangle$, where $\mathbb{TH}$ is the set of all nonempty Γ-theories, $\Sigma \cdot \Pi = \{\psi : \exists\theta \in \Pi(\theta \rightarrow \psi \in \Sigma)\}$, and $V(p) = \{\Delta \in \mathbb{TH} : p \in \Delta\}$.

It is straightforward but tedious to confirm that $\mathfrak{M}^\Gamma$ is in fact a model satisfying the requisite frame conditions. The proof of the truth lemma ($\mathfrak{M}^\Gamma \models_\Delta \varphi$ iff $\varphi \in \Delta$) is the same except for the added case of f, which we briefly review. Suppose $\mathsf{f} \in \Delta$; since Γ is normal, $\mathsf{f} \notin \Gamma$, hence $\Delta \not\subseteq \Gamma$ and so $\mathfrak{M}^\Gamma \models_\Delta \mathsf{f}$. Conversely, suppose $\mathsf{f} \notin \Delta$ and consider any $\varphi \in \Delta$; we will show $\varphi \in \Gamma$. Since Γ is regular, $\varphi \vee (\varphi \rightarrow \mathsf{f}) \in \Gamma$; since Γ is prime, either $\varphi \in \Gamma$, in which case we are done, or $\varphi \rightarrow \mathsf{f} \in \Gamma$, in which case $\mathsf{f} \in \Gamma \cdot \Delta = \Delta$, a contradiction. Thus, $\Delta \subseteq \Gamma$, whence $\mathfrak{M}^\Gamma \not\models_\Delta \mathsf{f}$, as desired. Completeness for $\mathsf{RM0}^{\mathsf{f}}$ follows in the usual manner, and essentially the same argument establishes completeness for J^{f} and $\mathsf{RM0}^{\mathsf{f,t}}$, a point to which we will return momentarily.

[41] Similarly, $\{\varphi : \exists\varphi_1, \ldots, \varphi_n \in \Delta((\varphi_1 \wedge \ldots \wedge \varphi_n) \rightarrow \varphi \in \Gamma)\}$, the smallest Γ-theory containing Δ, replaces the notion of the smallest L-theory containing Δ.

Using a well-known translation scheme (see Meyer (1966, pp. 198ff.)), $\mathsf{J}^{\perp}$ can be interpreted in $\mathsf{RM0}^{\perp,\mathsf{t}}$. Define the translation ${}^{\tau} : \mathcal{L}^{\perp} \to \mathcal{L}^{\perp,\mathsf{t}}$ as follows:

$p.\ p^{\tau} = p;$
$\perp.\ \perp^{\tau} = \perp;$
$\wedge.\ (\varphi \wedge \psi)^{\tau} = \varphi^{\tau} \wedge \psi^{\tau};$
$\vee.\ (\varphi \vee \psi)^{\tau} = \varphi^{\tau} \vee \psi^{\tau};$
$\to.\ (\varphi \to \psi)^{\tau} = (\varphi^{\tau} \wedge \mathsf{t}) \to \psi^{\tau}.$

Weiss (2022, §3.4) uses the model theory developed above, in part, to prove that this embedding of $\mathsf{J}^{\perp}$ in $\mathsf{RM0}^{\perp,\mathsf{t}}$ is exact:

Theorem 13 *For any* $\varphi \in \mathcal{L}^{\perp}$, $\vdash_{\mathsf{J}^{\perp}} \varphi$ *if and only if* $\vdash_{\mathsf{RM0}^{\perp,\mathsf{t}}} \varphi^{\tau}$.

It is worth remarking that Dunn and Meyer (1971, §3) use the same translation to exactly interpret LC (Gödel-Dummett logic; see Dummett (1959)) in RM.[42] In this way, it emerges that LC stands to RM roughly as J stands to $\mathsf{RM0}$.

Let us now briefly consider the case of J^{f} and $\mathsf{RM0}^{\mathsf{f},\mathsf{t}}$. Let the translation ${}^{\sigma} : \mathcal{L}^{\mathsf{f}} \to \mathcal{L}^{\mathsf{f},\mathsf{t}}$ be defined just like τ except:

$\mathsf{f}.\ \mathsf{f}^{\sigma} = \mathsf{f}.$

Essentially the same reasoning suffices to show that J^{f} can be interpreted exactly in $\mathsf{RM0}^{\mathsf{f},\mathsf{t}}$:

Theorem 14 *For any* $\varphi \in \mathcal{L}^{\mathsf{f}}$, $\vdash_{\mathsf{J}^{\mathsf{f}}} \varphi$ *if and only if* $\vdash_{\mathsf{RM0}^{\mathsf{f},\mathsf{t}}} \varphi^{\sigma}$.

Theorem 14 might be compared with results Meyer (1973, §1) has obtained to the effect that R is a conservative extension of J^{f}. (Note that Meyer (1973) formulates R in an unusually rich language.)

4 Proof Theory

In Sect. 2, we presented our logics using axiom systems, because they permit an easy, uniform presentation of the systems. For proof-theoretic work, it is better to use alternative proof systems. There are a variety of proof systems for relevant logics, but we

[42] RM (R-Mingle), the much better-known cousin of $\mathsf{RM0}$, is obtained by extending R by the mingle axiom (Ax3) *in the language with De Morgan negation*. Note, carefully, that $\mathsf{RM0}^{+}$ is *not* the positive fragment of RM (i.e., RM does not conservatively extend $\mathsf{RM0}^{+}$). RM has been exhaustively investigated in Dunn (1970, 1976), Anderson and Belnap (1975, §§29.3–5), and elsewhere. Technically, Dunn and Meyer (1971, §3) prove their translation result for Gödel-Dummett logic with respect to a system $\mathsf{RM}^{\perp,\mathsf{t}}$. Avron (1986), in effect, shows that t is dispensable for their result if the translation is modified so that $(\varphi \to \psi)^{\tau} = (\varphi^{\tau} \to \psi^{\tau}) \vee \psi^{\tau}$.

will not have space for presenting all of them here.[43] Instead, we will focus on a form of natural deduction that uses indexed formulas. We begin with Charlwood's natural deduction system for S in Sect. 4.1. Then, in Sect. 4.2, we turn to variations on Charlwood's system that accommodate J and (a fragment of) RM0, and also briefly examine Tennant's alternative approach to constructive relevant logic.

4.1 Natural Deduction for S

In this section, we examine a natural deduction system for relevant logic that operates on formulas indexed by finite (possibly empty) sets of natural numbers. These numbers, in essence, track the assumptions used in obtaining the formula in a derivation. Fitch-systems for relevant logics that operate on indexed formulas were introduced by Anderson (1960), further developed by Anderson and Belnap (1975), and generalized by Brady (1984). Rather than Fitch-systems, we will focus on tree-style (or Gentzen-Prawitz-style) natural deduction systems, to facilitate comparison with intuitionistic logic.[44] Tree-style systems with indexed formulas were introduced by Anderson (1978, pp. 5–6).[45] The rules for Charlwood's system for S—NS—are as follows.

$$\dfrac{\begin{array}{c}\dfrac{}{\varphi_{\{k\}}}\,(k)\\ \Pi \\ \psi_\alpha\end{array}}{\varphi \to \psi_{\alpha-\{k\}}}\,(\to I, k) \qquad \dfrac{\varphi \to \psi_\beta \qquad \varphi_\alpha}{\psi_{\alpha\cup\beta}}\,(\to E)$$

$$\dfrac{\varphi_\alpha \qquad \psi_\alpha}{\varphi \wedge \psi_\alpha}\,(\wedge I)$$

$$\dfrac{\varphi \wedge \psi_\alpha}{\varphi/\psi_\alpha}\,(\wedge E) \qquad \dfrac{\varphi/\psi_\alpha}{\varphi \vee \psi_\alpha}\,(\vee I)$$

$$\dfrac{\varphi \vee \psi_\alpha \qquad \begin{array}{c}\dfrac{}{\varphi_\alpha}\,(i)\\ \Pi_1 \\ \theta_{\alpha\cup\beta}\end{array} \qquad \begin{array}{c}\dfrac{}{\psi_\alpha}\,(j)\\ \Pi_2 \\ \theta_{\alpha\cup\beta}\end{array}}{\theta_{\alpha\cup\beta}}\,(\vee E, i, j)$$

In addition, there is a rule of assumption that permits one to assume a formula φ_α The $(\to I)$ rule requires that $k \in \alpha$, and it can only discharge assumptions where the index is a singleton. If $(\to I)$ discharges an assumption of $\varphi_{\{k\}}$, all occurrences of assumptions of $\varphi_{\{k\}}$ above that application must be discharged. Finally, in an application of $(\vee E)$ with major premise $\varphi \vee \psi_\alpha$, all occurrences of assumptions of φ_α above the first minor premise and all occurrences of assumptions of ψ_α above the

[43] Read (1988), Slaney (1990), Restall (2000), and Bimbó (2014) provide good overviews of different sequent and natural deduction systems for relevant logics.

[44] Standefer (2018, 2021b) discusses connections between Fitch(-Jaśkowski)-style natural deduction systems with indexed formulas and tree-style natural deduction systems.

[45] A sequent-style system with indexed formulas for S(→) was introduced as early as Urquhart (1971). For further developments along these and related lines, consult Urquhart (1973, §8), Giambrone and Urquhart (1987), Kashima (2003), and Weiss (2021b, §6).

second minor premise are discharged by $(\vee E)$. *Derivations* in NS are trees defined in the manner of Prawitz (1965), subject to the constraints on discharging assumptions described above.

There are two important, related differences between this system and the natural deduction systems studied by Anderson, Belnap, and Brady. The first difference is the $(\vee E)$ rule. The rule used in the Anderson-Belnap-Brady systems is the following.[46]

$$\frac{\varphi \vee \psi_\alpha \qquad \varphi \to \theta_\beta \qquad \psi \to \theta_\beta}{\theta_{\alpha\cup\beta}}\ (\vee E^*)$$

In this rule, the minor premises are conditionals. This violates a commonly adopted requirement on proof rules, namely that they do not display any connectives apart from the one governed by that rule.[47] Additionally, the use of the conditional here obscures an important feature that can be brought out if the rules are rewritten, following Urquhart (1989), in a way that avoids the conditionals:

$$\frac{\varphi \vee \psi_\alpha \qquad \begin{array}{c}\overline{\varphi_{\{k_1\}}}\ (\mathrm{i})\\ \Pi_1\\ \theta_{\{k_1\}\cup\beta}\end{array} \qquad \begin{array}{c}\overline{\psi_{\{k_2\}}}\ (\mathrm{j})\\ \Pi_2\\ \theta_{\{k_2\}\cup\beta}\end{array}}{\theta_{\alpha\cup(\beta\setminus\{k_1,k_2\})}}\ (\vee E^{**}, i, j)$$

This rule trades conditionals for subproofs, and in these subproofs, the assumptions of φ and ψ are required to have new singleton indices. Let us call the assumptions discharged by $\vee(E^{**})$ the *minor premise assumptions*. The indices track which assumptions are used where in a proof, and $(\vee(E^{**})$ has the effect that the minor premise assumptions do not depend on the same assumptions used in deriving the major premise disjunction. Consequently, distribution, $(\varphi \wedge (\psi \vee \theta)) \to ((\varphi \wedge \psi) \vee (\varphi \wedge \theta))$, is not derivable using the $\vee(E^{**})$ rule, or equivalently the $\vee(E^*)$ rule, so the Anderson-Belnap-Brady systems require a primitive rule of distribution:

$$\frac{\varphi \wedge (\psi \vee \theta)_\alpha}{(\varphi \wedge \psi) \vee (\varphi \wedge \theta)_\alpha}\ (\mathit{Dist})$$

This is, as Urquhart (1989, pp. 168-169) notes, unsatisfactory, particularly if one takes the introduction and elimination rules to in some sense give the meaning of the connectives. It is also unsatisfactory from a purely formal point of view, as the inclusion of the (*Dist*) rule prevents a proof of a normalization theorem for these systems.

In the rule $(\vee E)$, on the other hand, the minor premise disjuncts have the same subscript as the major premise disjunction, so they depend on the same assumptions. To illustrate how this relates to distribution, it is worth looking at a derivation.

[46] See Anderson and Belnap (1975, pp. 346ff.) and Brady (1984).

[47] This is separability, as described by Wansing (1998, p. 8).

$$
\begin{prooftree}
\AxiomC{$\varphi \wedge (\psi \vee \theta)_{\{1\}}$}
\UnaryInfC{$\psi \vee \theta_{\{1\}}$}
\AxiomC{$\varphi \wedge (\psi \vee \theta)_{\{1\}}$}
\UnaryInfC{$\varphi_{\{1\}}$}
\AxiomC{}
\UnaryInfC{$\psi_{\{1\}}$}
\RightLabel{$(\wedge I)$}
\BinaryInfC{$\varphi \wedge \psi_{\{1\}}$}
\RightLabel{$(\vee I)$}
\UnaryInfC{$(\varphi \wedge \psi) \vee (\varphi \wedge \theta)_{\{1\}}$}
\AxiomC{$\varphi \wedge (\psi \vee \theta)_{\{1\}}$}
\UnaryInfC{$\varphi_{\{1\}}$}
\AxiomC{}
\UnaryInfC{$\theta_{\{1\}}$}
\RightLabel{$(\wedge I)$}
\BinaryInfC{$\varphi \wedge \theta_{\{1\}}$}
\RightLabel{$(\vee I)$}
\UnaryInfC{$(\varphi \wedge \psi) \vee (\varphi \wedge \theta)_{\{1\}}$}
\RightLabel{$(\vee E)$}
\TrinaryInfC{$(\varphi \wedge \psi) \vee (\varphi \wedge \theta)_{\{1\}}$}
\end{prooftree}
$$

This derivation is not available with the $(\vee E^{**})$ rule, because the minor premise disjunctions would be required to have different subscripts (e.g., $\psi_{\{2\}}$ and $\theta_{\{3\}}$), which would prevent the use of $(\wedge I)$ in the minor premise subproofs. Stepping back, the use of the $(\vee E)$ rule, rather than $(\vee E^{*})$ or $(\vee E^{**})$, permits one to prove a normalization theorem, which has many important proof-theoretic consequences, some of which we will examine below. Formally, this means that Charlwood's natural deduction system is proof-theoretically nice in ways similar to those of Prawitz's (1965) natural deduction system for J. Philosophically, it suggests that the proper elimination rule for disjunction is $(\vee E)$, rather than $(\vee E^{*})$ or $(\vee E^{**})$.[48]

We write $\vdash_{\mathsf{NS}} \varphi$ to indicate that $\varphi_{\emptyset}$ has a derivation in Charlwood's natural deduction system NS in which all assumptions have been discharged. Charlwood (1978, 1981) proves the following completeness result.

Theorem 15 $\vdash_{\mathsf{NS}} \varphi$ *iff* $\vdash_{\mathsf{S}} \varphi$ *iff* $\models_{\mathsf{S}} \varphi$.

To prove this, Charlwood shows that NS is theorem-equivalent to an axiomatic formulation of S.[49] Then, he uses a canonical model construction to show that S is sound and complete with respect to the semilattice semantics.

Let us now examine some of the elegant proof-theoretic features of NS. The first result of Charlwood's that we will highlight requires a definition.

Definition 11 A sequence of formula occurrences of $\varphi^1_\alpha, \ldots, \varphi^n_\alpha$ in a proof is a *detour sequence* iff φ^1_α is the conclusion of an *I*-rule and for $1 \leq i < n$, φ^i_α is a minor premise of an $\vee(E)$ rule and φ^{i+1}_α is the conclusion of the $\vee(E)$ rule, and φ^n_α is the premise of an E rule $\vee(E)$. A proof with no detour sequences is called a *normal proof*.

A detour sequence of length 1 can be removed by using one of the reduction steps in Fig. 1. In the $\rightarrow$-reduction, $\Pi_1^{[k/\beta]}$ is Π_1 with all occurrences of *k* in subscripts replaced by the members of β.

A detour sequence of length > 1 can be shortened using the permutation reduction in Fig. 2. In this reduction, Π_4 is one or two subproofs, as appropriate for the number of minor premises required by (*R*).

[48] This point has also been emphasized by Urquhart (1989, p. 169), who writes, "Thus I think it not unreasonable to claim that the rules given above [$(\vee I)$ and $(\vee E)$] constitute the correct rules for disjunction in relevant implication".

[49] The proof of this equivalence, which we omit, has not been published but can be found in Charlwood (1978).

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \varphi^1_\alpha\end{matrix} \qquad \begin{matrix}\Pi_2\\ \varphi^2_\alpha\end{matrix}}{\varphi^1 \wedge \varphi^2_\alpha}\,(\wedge I)}{\varphi^i_\alpha}\,(\wedge E) \qquad\qquad \begin{matrix}\Pi_i\\ \varphi^i_\alpha\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\dfrac{}{\varphi_{\{k\}}}\,(\mathrm{k})\\ \Pi_1\\ \psi_\alpha\end{matrix}}{\varphi \to \psi_{\alpha\setminus\{k\}}}\,(\to I),\,(\mathrm{k}) \qquad \begin{matrix}\Pi_2\\ \varphi_\beta\end{matrix}}{\psi_{(\alpha\setminus\{k\})\cup\beta}}\,(\to E) \qquad\qquad \begin{matrix}\Pi_2\\ \varphi_\beta\\ \Pi_1^{[k/\beta]}\\ \psi_{(\alpha\setminus\{k\})\cup\beta}\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_3\\ \varphi^i_\alpha\end{matrix}}{\varphi^1 \vee \varphi^2_\alpha} \qquad \begin{matrix}\overline{\varphi^1_\alpha}\\ \Pi_1\\ \theta_{\alpha\cup\beta}\end{matrix} \qquad \begin{matrix}\overline{\varphi^2_\alpha}\\ \Pi_2\\ \theta_{\alpha\cup\beta}\end{matrix}}{\theta_{\alpha\cup\beta}}\,(\vee E) \qquad\qquad \begin{matrix}\Pi_3\\ \varphi^i_\alpha\\ \Pi_i\\ \theta_{\alpha\cup\beta}\end{matrix}$$

Fig. 1 Detour reductions

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \varphi \vee \psi_\alpha\end{matrix} \quad \begin{matrix}\Pi_2\\ \theta_{\alpha\cup\beta}\end{matrix} \quad \begin{matrix}\Pi_3\\ \theta_{\alpha\cup\beta}\end{matrix}}{\theta_{\alpha\cup\beta}}\,(\vee E) \qquad \Pi_4}{\delta_{\alpha\cup\beta\cup\gamma}}\,(R) \qquad \dfrac{\begin{matrix}\Pi_1\\ \varphi \vee \psi_\alpha\end{matrix} \quad \dfrac{\begin{matrix}\Pi_2\\ \theta_{\alpha\cup\beta}\end{matrix} \quad \Pi_4}{\delta_{\alpha\cup\beta\cup\gamma}}\,(R) \quad \dfrac{\begin{matrix}\Pi_3\\ \theta_{\alpha\cup\beta}\end{matrix} \quad \Pi_4}{\delta_{\alpha\cup\beta\cup\gamma}}\,(R)}{\delta_{\alpha\cup\beta\cup\gamma}}\,(\vee E)$$

Fig. 2 Permutation reduction

Charlwood (1978) proves a normalization theorem, showing that every proof of $\varphi_\emptyset$ can be converted into a normal proof of $\varphi_\emptyset$ using the reductions described above.

Theorem 16 *If* $\vdash_{\mathsf{NS}} \varphi$*, then there is a normal proof ending in* $\varphi_\emptyset$.

This is proved using a proof technique similar to the normalization theorem proved by Prawitz (1965). In particular, proofs are assigned pairs (d, n) where d is the greatest complexity of the first formula in the sequence plus the length of the sequence and n is the number of detour sequences of complexity d. The normalization procedure eliminates a maximally complex detour sequence with no equally complex detour sequences above it. The procedure then proceeds to another maximally complex detour sequence, decreasing the second component of the pair until the first component drops. This continues until the proof is assigned (0, 0), which is a normal proof.

This is a proof of weak normalization, as there are restrictions on the order in which detours can be removed. The question of strong normalization, removing detours in any order, is open. Given the similarities between the proof of weak normalization for NS and the proof of weak normalization for J, we expect that proof strategies for strong normalization for the latter can be adapted to the former, although we leave this for future work.

Many of the consequences of normalization for intuitionistic natural deduction can also be obtained for NS. The first we consider is the subformula property.

Theorem 17 *In a normal proof Π in* NS *ending in $\varphi_\emptyset$, all formulas occurring in Π are subformulas of φ.*

We note that there will be natural numbers occurring in subscripts in the proof that do not occur in the subscript of the conclusion, as that is empty. Nonetheless, there is a bound on the number of subscripts with distinct single numbers that can occur in the proof, as each such must be discharged by $(\to I)$. While formulas with singleton subscripts can be discharged by $(\vee E)$, any such subscripts must be introduced by other assumption steps that are discharged by applications of $(\to I)$.

A consequence of the subformula property, which is important for our purposes, is the disjunction property.

Theorem 18 NS *enjoys the disjunction property.*

Thus, NS (and so S; see Theorem 15) realizes one of the most characteristic features of constructive logic. It also realizes one of the most characteristic features of relevant logic, since Weiss (2019) proved (semantically) that S satisfies the variable-sharing property. It is a relatively strong logic, as it properly extends R. Besides the example mentioned in Sect. 2, two further examples of formulas which are derivable in S but not R are[50]:

- $((\varphi \to (\psi \vee \theta)) \wedge (\psi \to \theta)) \to (\varphi \to \theta)$, and
- $(\varphi \to ((\varphi \to \varphi) \vee \varphi)) \to (\varphi \to \varphi)$.

While $\mathsf{RM0}$ is also a proper extension of R, as mentioned above, it is not known whether $\mathsf{S} \subseteq \mathsf{RM0}$. The converse containment does not hold, as (Ax3) is invalid in S.

Continuing with our proof-theoretic investigation of S, some further definitions are required (cf. Harrop (1960)).

Definition 12 The set of *Harrop formulas* is defined inductively as follows. All atoms are Harrop formulas. If ψ and θ are Harrop formulas, then so are $\psi \wedge \theta$ and $\varphi \to \psi$, for any formula φ. Nothing else is a Harrop formula.

Definition 13 A logic L has the *Harrop property* iff whenever $\vdash_{\mathsf{L}} \varphi \to (\psi \vee \theta)$ and φ is a Harrop formula, then $\vdash_{\mathsf{L}} \varphi \to \psi$ or $\vdash_{\mathsf{L}} \varphi \to \theta$.

Definition 14 *Mints' rule* is admissible for a logic L iff $\vdash_{\mathsf{L}} (\varphi \to \psi) \to (\varphi \vee \theta)$ only if $\vdash_{\mathsf{L}} ((\varphi \to \psi) \to \varphi) \vee ((\varphi \to \psi) \to \theta)$.

[50] The first of these is a variant of the example due to Dunn and Meyer, as reported by Urquhart (1972, p. 163). The second is from Standefer (2021a, p. 244).

Prawitz (1965, p. 55) shows that J has the Harrop property, and Mints' rule is admissible for J.[51] As we will see, Mints' rule is admissible in S, and S also enjoys the Harrop property. To show these facts, we will need to state another definition.

Definition 15 A sequence $\langle \psi^1_{\alpha_1}, \ldots, \psi^n_{\alpha_n} \rangle$ is a *track* in a proof Π iff $\psi^1_{\alpha_1}$ is an assumption, $\psi^i_{\alpha_i}$ is immediately above $\psi^{i+1}_{\alpha_{i+1}}$, and $\psi^n_{\alpha_n}$ is either the minor premise of an application of a rule or the conclusion of the proof. A *main track* is any track whose final member is the conclusion of the proof.

One can show that in a normal proof, each track splits into two parts: an *E*-part, where $\psi^i_{\alpha_i}$ is the premise of an *E*-rule that has $\psi^{i+1}_{\alpha_{i+1}}$ as the conclusion, followed by an *I*-part where each $\psi^i_{\alpha_i}$ is the premise of an *I*-rule that has $\psi^{i+1}_{\alpha_{i+1}}$ as its conclusion. One of the *E*- and *I*-parts may be empty.

Theorem 19 *Mints' rule is admissible in* S.

Proof Suppose that there is a proof of $(\varphi \to \psi) \to (\varphi \vee \theta)_\emptyset$ in NS. Then there is a normal proof, so we can consider the final rule used, which must be $(\to I)$. So, there is a proof of $\varphi \vee \theta_{\{k\}}$ from $\varphi \to \psi_{\{k\}}$ (for some *k*). Let us consider a main track running from the conclusion to $\varphi \to \psi_{\{k\}}$. The formula $\varphi \vee \theta_{\{k\}}$ must be on the introduction portion of the track, so the rule it is the conclusion of must be $(\vee I)$, with the premise being either $\varphi_{\{k\}}$ or $\theta_{\{k\}}$. Suppose it is $\varphi_{\{k\}}$. In that case, $(\varphi \to \psi) \to \varphi_\emptyset$ can be derived via $(\to I)$, from which the desired conclusion follows via $(\vee I)$. The case where $\theta_{\{k\}}$ is the premise is similar.

A similar proof establishes that S has the Harrop property.

Theorem 20 S *enjoys the Harrop property.*

Proof Suppose that there is a proof of $\varphi \to (\psi \vee \theta)_\emptyset$, where φ is a Harrop formula, in NS. Then there is a normal proof. The final rule used in the proof must be $(\to I)$, so there is a proof of $\psi \vee \theta_{\{k\}}$ from $\varphi_{\{k\}}$ (for some *k*). Consider a main track containing the conclusion. From the assumptions, since φ is a Harrop formula, $\psi \vee \theta_{\{k\}}$ must be on the introduction portion of the track. The only rule it can be the conclusion of is $(\vee I)$, so the premise of that rule must be $\psi_{\{k\}}$ or $\theta_{\{k\}}$. In the former case, $\varphi \to \psi_\emptyset$ is derivable, and in the latter case, $\varphi \to \theta_\emptyset$ is derivable.

[51] Logan (2021) used algebraic methods to prove that the weak relevant logic, B has the Harrop property, provided that φ is an atom.

As far as we know, there has not been any previous study of admissible rules for S. This is unfortunate, since S has some well-behaved proof systems and there are many similarities between J and S.[52]

4.2 Variations

Natural deduction systems for intuitionistic logic can be obtained in at least two ways from Charlwood's system NS. One option is to remove the indices and drop the side condition on $(\rightarrow I)$. The result, when the appropriate rule for $\perp$ is added, is the natural deduction system for J given by Prawitz (1965).[53] A distinct option, more in keeping with Charlwood's presentation, is to instead *add* the following rules to NS:

$$\frac{\mathfrak{a}_\alpha}{\mathfrak{a}_{\alpha\cup\beta}}\ (\mathfrak{a}H) \qquad\qquad \frac{\perp_\alpha}{\mathfrak{a}_\alpha}\ (\perp E)$$

In these rules, $\mathfrak{a}$ is either of the form p or $\perp$. Semantically, $(\mathfrak{a}H)$ corresponds to the heredity condition and $(\perp E)$ to the explosion condition imposed in Definition 6. We call the system so-constituted NJ ($\vdash_{\mathsf{NJ}}$ is defined as above). It can be shown by induction that generalizations of these additional rules are derivable in NJ.

Theorem 21 *The following rules are derivable in* NJ.

$$\frac{\varphi_\alpha}{\varphi_{\alpha\cup\beta}}\ (H) \qquad\qquad \frac{\perp_\alpha}{\varphi_\alpha}\ (\perp E')$$

Proof The proof is by induction on the structure of φ. We will present two inductive cases for the proof of (H).

Case: φ is $\psi \vee \theta$.

$$\dfrac{\psi \vee \theta_\alpha \qquad \dfrac{\dfrac{\overline{\psi_\alpha}}{\psi_{\alpha\cup\beta}}\ (H)}{\psi \vee \theta_{\alpha\cup\beta}}\ (\vee I) \qquad \dfrac{\dfrac{\overline{\theta_\alpha}}{\theta_{\alpha\cup\beta}}\ (H)}{\psi \vee \theta_{\alpha\cup\beta}}\ (\vee I)}{\psi \vee \theta_{\alpha\cup\beta}}\ (\vee E)$$

Case: φ is $\psi \rightarrow \theta$. We assume $k \notin \alpha \cup \beta$.

[52] There are well-behaved proof systems for many other relevant logics without De Morgan negation, as illustrated, for example, by Restall (2000). We think that the study of admissible rules for these logics could be interesting as well.

[53] We note that Prawitz (1965, ch. 7) gave a natural deduction system without indices that is equivalent to Charlwood's NS (see Charlwood (1978, pp. 49–52)). Prawitz's system uses side conditions on $(\rightarrow I)$ and on $(\vee E)$ that have the same effects as the indices.

$$
\begin{prooftree}
\AxiomC{$\psi \to \theta_\alpha$}
\AxiomC{}
\UnaryInfC{$\psi_{\{k\}}$}
\RightLabel{$(\to E)$}
\BinaryInfC{$\theta_{\alpha\cup\{k\}}$}
\RightLabel{(H)}
\UnaryInfC{$\theta_{\alpha\cup\beta\cup\{k\}}$}
\RightLabel{$(\to I)$}
\UnaryInfC{$\psi \to \theta_{\alpha\cup\beta}$}
\end{prooftree}
$$

The remaining cases for (*H*) are immediate from the inductive hypothesis.

For $(\bot E')$, the cases are immediate from the inductive hypothesis apart from the $\psi \to \theta$ case, which we present here. We assume $k \notin \alpha$.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\psi_{\{k\}}$}
\RightLabel{(H)}
\UnaryInfC{$\psi_{\alpha\cup\{k\}}$}
\AxiomC{$\bot_\alpha$}
\RightLabel{$(\bot E')$}
\UnaryInfC{θ_α}
\RightLabel{(H)}
\UnaryInfC{$\theta_{\alpha\cup\{k\}}$}
\RightLabel{$(\wedge I)$}
\BinaryInfC{$\psi \wedge \theta_{\alpha\cup\{k\}}$}
\RightLabel{$(\wedge E)$}
\UnaryInfC{$\theta_{\alpha\cup\{k\}}$}
\RightLabel{$(\to I)$}
\UnaryInfC{$\psi \to \theta_\alpha$}
\end{prooftree}
$$

We note that the generalized forms of the two rules may introduce detours. Our focus is not on proof-theoretic features of NJ, so we will not attempt to modify the rules to avoid this feature.

Completeness for NJ with respect to theoremhood in J is straightforward. The easiest way to see this is via a derivation of (Ax2) in NJ.

Lemma 5 *(Ax2) is derivable in* NJ.

Proof Partly just to further illustrate how derivations work, we first prove something more elaborate, namely, $((\varphi \wedge \psi) \to \theta) \to (\varphi \to (\psi \to \theta))$.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$((\varphi \wedge \psi) \to \theta)_{\{1\}}$}
\AxiomC{}
\UnaryInfC{$\varphi_{\{2\}}$}
\RightLabel{(H)}
\UnaryInfC{$\varphi_{\{2,3\}}$}
\AxiomC{}
\UnaryInfC{$\psi_{\{3\}}$}
\RightLabel{(H)}
\UnaryInfC{$\psi_{\{2,3\}}$}
\RightLabel{$(\wedge I)$}
\BinaryInfC{$\varphi \wedge \psi_{\{2,3\}}$}
\RightLabel{$(\to E)$}
\BinaryInfC{$\theta_{\{1,2,3\}}$}
\RightLabel{$(\to I)$}
\UnaryInfC{$\psi \to \theta_{\{1,2\}}$}
\RightLabel{$(\to I)$}
\UnaryInfC{$\varphi \to (\psi \to \theta)_{\{1\}}$}
\RightLabel{$(\to I)$}
\UnaryInfC{$((\varphi \wedge \psi) \to \theta) \to (\varphi \to (\psi \to \theta))_\emptyset$}
\end{prooftree}
$$

As a special case, we have that $((\varphi \wedge \psi) \to \varphi) \to (\varphi \to (\psi \to \varphi))_\emptyset$ is derivable. $(\varphi \wedge \psi) \to \varphi_\emptyset$ is derivable as well. Combining these two proofs and using $(\to E)$ results in $\varphi \to (\psi \to \varphi)_\emptyset$, as desired.

We have the following theorem as a corollary of the lemma.

Theorem 22 *If* $\vdash_{\mathsf{J}} \varphi$*, then* $\vdash_{\mathsf{NJ}} \varphi$.

Proof Since NS is complete with respect to theorems of S, we can restrict our attention to (Ax2) and (Ax13). Lemma 5 shows that (Ax2) is derivable, and (Ax13) is immediate using $(\bot E')$. Closure of the set of theorems of NJ under (Ru1) and (Ru2) is guaranteed by $(\to E)$ and $(\wedge I)$, which suffices.

We prove soundness by providing a means of interpreting NJ in the semantics for J from Sect. 3.1. Observe that this is not entirely straightforward because NJ does not operate on formulas, but on *indexed* formulas. We circumvent this difficulty by employing the notion of an interpretation function (cf. Giambrone and Urquhart (1987, p. 438)).[54] An *interpretation* in a semilattice model $\mathfrak{M} = \langle S, 0, \sqcup, V\rangle$ is a function $f : \mathbb{Z}^+ \to S$. We say that a model $\mathfrak{M}$ and an interpretation *f* in it *satisfy* an indexed formula φ_α if $\mathfrak{M} \models_{\bigsqcup_{i\in\alpha} f(i)} \varphi$.

Lemma 6 *For every interpretation f in every* J *semilattice model* $\mathfrak{M} = \langle S, 0, \sqcup, V, \rangle$ *if* $\mathfrak{M}$ *and f satisfy every undischarged assumption* φ_α *of an* NJ *constructed tree* $\mathcal{T}$ *with root* ψ_β*, then* $\mathfrak{M}$ *and f satisfy* ψ_β*.*

Proof The argument proceeds by induction on the length of proof by considering cases of how ψ_β is introduced. We just examine the cases of $(\to I)$, $(\to E)$, $(\vee E)$, and $(\bot E)$.

Ad $(\to I)$: Suppose for contradiction that some *f* and $\mathfrak{M} = \langle S, 0, \sqcup, V\rangle$ satisfy all the undischarged assumptions of a tree $\mathcal{T}$ but not its root, $\varphi \to \psi_{\alpha-\{k\}}$, which was introduced by $(\to I)$. Then $\mathfrak{M} \not\models_{\bigsqcup_{i\in\alpha-\{k\}} f(i)} \varphi \to \psi$, whence for some $s \in S$, $\mathfrak{M} \models_s \varphi$ but $\mathfrak{M} \not\models_{(\bigsqcup_{i\in\alpha-\{k\}} f(i))\sqcup s} \psi$. Put $g = f$ except $g(k) = s$. It is clear that $\mathfrak{M}$ and *g* satisfy all the undischarged assumptions but not the root of the subtree $\mathcal{T}'$ of $\mathcal{T}$ terminating in ψ_α, contradicting the induction hypothesis.

Ad $(\to E)$: Suppose for contradiction that some *f* and $\mathfrak{M} = \langle S, 0, \sqcup, V\rangle$ satisfy all the undischarged assumptions of a tree $\mathcal{T}$ but not its root, $\psi_{\alpha\cup\beta}$, which was introduced by $(\to E)$. Clearly, $\mathfrak{M}$ and *f* satisfy all the undischarged assumptions of the subtrees $\mathcal{T}'$ and $\mathcal{T}''$ with roots $\varphi \to \psi_\beta$ and φ_α, whence by the induction hypothesis, $\mathfrak{M} \models_{\bigsqcup_{i\in\beta} f(i)} \varphi \to \psi$ and $\mathfrak{M} \models_{\bigsqcup_{i\in\alpha} f(i)} \varphi$. Therefore, $\mathfrak{M} \models_{(\bigsqcup_{i\in\alpha} f(i))\sqcup(\bigsqcup_{i\in\beta} f(i))} \psi$, that is, $\mathfrak{M} \models_{\bigsqcup_{i\in\alpha\cup\beta} f(i)} \psi$, contradicting the assumption.

Ad $(\vee E)$: Suppose for contradiction that some *f* and $\mathfrak{M} = \langle S, 0, \sqcup, V\rangle$ satisfy all the undischarged assumptions of a tree $\mathcal{T}$ but not its root, $\theta_{\alpha\cup\beta}$, which was the conclusion of $(\vee E)$. $\mathfrak{M}$ and *f* satisfy all the undischarged assumptions of the subtree $\mathcal{T}'$ with root $\varphi \vee \psi_\alpha$ and all the undischarged assumptions, apart from those discharged by $(\vee E)$, of the minor premise subproofs $\mathcal{T}''$ and $\mathcal{T}'''$ both of whose roots are occurrences of $\theta_{\alpha\cup\beta}$. By the induction hypothesis, $\mathfrak{M} \models_{\bigsqcup_{i\in\alpha} f(i)} \varphi \vee \psi$, so $\mathfrak{M} \models_{\bigsqcup_{i\in\alpha} f(i)} \varphi$ or $\mathfrak{M} \models_{\bigsqcup_{i\in\alpha} f(i)} \psi$. Without loss of generality, we consider the

[54] Urquhart (1973, p. 32) employs a similar device to prove soundness for an indexed sequent calculus and Weiss (2023, §5) adapts the idea to prove soundness for indexed tableaux. Charlwood (1978, pp. 29–35) pursues a purely syntactic soundness strategy.

former case. $\mathfrak{M}$ and f satisfy all the undischarged assumptions of $\mathcal{T}''$, so by the inductive hypothesis, $\mathfrak{M} \models_{\bigsqcup_{i\in\alpha\cup\beta} f(i)} \theta$, which is a contradiction.

Ad $(\bot E)$: Suppose for contradiction that some f and $\mathfrak{M} = \langle S, 0, \sqcup, V\rangle$ satisfy all the undischarged assumptions of a tree $\mathcal{T}$ but not its root, $\mathfrak{a}_\alpha$, which was introduced by $(\bot E)$. By the induction hypothesis, $\mathfrak{M} \models_{\bigsqcup_{i\in\alpha} f(i)} \bot$, so by Definition 6, $\mathfrak{M} \models_{\bigsqcup_{i\in\alpha} f(i)} p$ for every p. So, clearly, $\mathfrak{M} \models_{\bigsqcup_{i\in\alpha} f(i)} \mathfrak{a}$, a contradiction.

Theorem 23 *If* $\vdash_{\mathsf{NJ}} \varphi$, *then* $\vdash_{\mathsf{J}} \varphi$.

Proof Suppose $\vdash_{\mathsf{NJ}} \varphi$. Then all assumptions are discharged in an NJ constructed tree $\mathcal{T}$ with root $\varphi_\emptyset$, so the hypotheses of Lemma 6 are vacuously satisfied and we may conclude that for every model $\mathfrak{M}$ and interpretation f, $\mathfrak{M} \models_{\bigsqcup_{i\in\emptyset} f(i)} \varphi$, that is, $\mathfrak{M} \models_0 \varphi$; thus, $\models_{\mathsf{J}} \varphi$, and so $\vdash_{\mathsf{J}} \varphi$ by Theorem 7.

A natural deduction system similar to NJ can be given for $\mathsf{RM0}(\to, \land)$ by adding the mingle heredity rule,

$$\frac{p_\alpha \qquad p_\beta}{p_{\alpha\cup\beta}}\ (pMH)$$

to the NS rules for the conditional and conjunction (for the semantic context, see footnote 29). The sequent-based natural deduction systems of Restall (2000) can be used to give a different sort of proof system for all of RM0. Investigation of such systems is, however, outside the scope of this paper.

While our focus has been on relevant logics in the tradition of Anderson and Belnap, we should mention one other proof-theoretically oriented approach to combining relevant and constructive logics. That approach is the core logic of Tennant (2017). Tennant's core logic, CT, is defined using a variation of the usual rules for intuitionistic logic. Some of the rules, notably $(\to I)$, have restrictions on discharging assumptions. In his system, $\bot$ is treated as a kind of structure that cannot be embedded, rather than a formula, so negation is not definable from it. The differences between CT and J only emerge in the presence of negation, and in particular when considering sets of inconsistent assumptions, so in the setting of this paper, CT and J will be the same in terms of derivability.

With negation in the language, differences emerge. Tennant shows that if $X \vdash_{\mathsf{J}} \varphi$, then there is some $Y \subseteq X$ such that either $Y \vdash_{\mathsf{CT}} \varphi$ or $Y \vdash_{\mathsf{CT}} \bot$. Further, he obtains a kind of variable-sharing property.

Theorem 24 *If there is a proof in* CT *with conclusion* φ *whose undischarged premises are all of those in the non-empty set* Γ, *then there is some shared atom in both* Γ *and* φ.

In terms of sequents, we can put the theorem as follows: If $X : \varphi$ has a proof, with $X \neq \emptyset$, then there is an atom shared between X and φ. This is variable-sharing at the level of the turnstile. CT does not have the variable-sharing property in the sense of

Definition 4 as $q \to (p \to p)$ is a theorem of CT. Further comparison between versions of relevant consequence, such as the bunched systems of Logan (2022) or the multiset-based proposal of Badia et al. (2024) seems likely to be illuminating.

5 Conclusions

In this paper, we have seen that constructivism and relevantism are not only compatible enterprises in philosophical logic, but have much to offer one another. On the one hand, we illustrated how adopting a constructivist perspective on negation can make available to the relevant logician systems like RM0 and S that simultaneously realize deductive strength, semantic and proof-theoretic elegance, and desiderata such as the disjunction property and the variable-sharing property. On the other hand, we indicated how constructivism might be enriched by relevant logic both by studying alternative systems that combine features of constructivism with relevance and by gaining new insights into intuitionistic logic itself (e.g., through the embedding results of Sect. 3.2 and the novel natural deduction system NJ of Sect. 4.2).

We expect that further study of the relations between relevant and constructive logics would yield dividends for both camps.[55] We mention a few possible specific avenues for further exploration. There is a rich constructivist tradition of systematically studying the lattice of so-called *intermediate logics*, that is, logics intermediate between intuitionistic and classical logic.[56] It would be of interest to see whether and how the results and techniques of this research program might be extended to a larger lattice rooted in a logic that is both constructive and relevant (say, RM0).

There is very little known about *quantified* constructive relevant logics, and even less is known about first-order constructive relevant theories. We hope that some of the tools that have been employed in the study of HA (Heyting Arithmetic) and other constructivist theories might be extended to constructive relevant theories of arithmetic. The connections between these techniques and metavaluations make us optimistic (cf. Sect. 2.2), but the details remain to be worked out.

Finally, we note that while the decision problem for S was recently solved in the negative by Knudstorp (2024), the decision problem for RM0 appears to still be open.[57] The related problem of whether RM0 has the finite model property appears to be open as well. However, it is known that certain fragments of RM0—for example, RM0($\to$)—possess the finite model property and are decidable (see Meyer and Ono (1994) and Meyer (2001)).

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[55] Incidentally, there has also been some study of both (broadly) relevant and intermediate logics—for example, LC and RM—from a fuzzy logic perspective (see, e.g., Cintula et al. (2011a, 2011b)), and further investigation of connections between fuzzy logics and relevant and constructive logics may bear fruit.

[56] Seminal work in this area includes Dummett and Lemmon (1959), Jankov (1968b), and Gabbay and De Jongh (1974).

[57] The wide-ranging undecidability results of Urquhart (1984) cover neither S nor RM0.

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Declarations

Conflict of interest Not applicable.

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