Universal necessity and deep classicality

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Abstract

The universal conception of necessity says that necessary truth is truth in all possible worlds. This idea is well studied in the context of classical possible worlds models, and there its logic is S5. The universal conception of necessity is less well studied in models for non-classical logics. We will present some preliminary results on universal necessity on models for intuitionistic logic, first-degree entailment, and relevant logics. We will close by discussing a way in which universal necessity is a very classical concept.

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1 Introduction

It is well-known that there are many different presentations of the classically-based modal logic S5. One way is via any of the common axiomatizations. Another is the use of relational frames $\langle W, S\rangle$ where the modal accessibility relation, S is an equivalence relation. Yet another is the use of relational frames with a universal modal accessibility relation, S, i.e. for all worlds $x, y \in W$, Sxy. In the setting of models on universal frames, the usual truth condition for necessity is equivalent to saying that a formula is true at a world iff it is true at all worlds. Because of this, we can, equally, consider frames without an accessibility relation, which will be useful in considering frames for non-classical logics.

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S5 is, in many ways, an interesting logic, and its distinctive features are brought out via the coincidence of these presentations.[1] Once one moves away from frames for classically-based modal logic, however, the situation changes. Standefer (2023) showed that in the setting of relevant logics, these three presentations yield at least three distinct logics, with some being incomparable.

The purpose of this paper is to further investigate the necessity operator of universal frames, what we will call *universal necessity*. We will consider universal necessity against the backdrop of three sorts of frames for non-classical logics: intuitionistic logic, first-degree entailment, and the relevant logic B. By studying universal necessity in these settings, we will uncover some commonalities to the logics of universal necessity. We will discuss this commonality in the final section, where we will argue that universal necessity is, in a sense, a deeply classical concept.

2 Intuitionistic Logic

The basic language for this paper is $\{\to, \wedge, \vee, \neg\}$. There will be a countable set of atoms, At and complex formulas will be constructed in the usual way. We will add a singulary connective, $\mathbb{U}$, for universal necessity, which will be the main topic of our investigations below. We will define the biconditional, $A \leftrightarrow B$ as $(A \to B) \wedge (B \to A)$. We will define three logics semantically using different classes of frames, so that L will be the set of formulas valid in the appropriate class frames.[2] For a logic L so presented, the logic $\mathsf{L}^{\mathbb{U}}$ will be the extension of L with $\mathbb{U}$, namely the set of formulas valid in the extended language in the appropriate class of frames.

In this section, we will study the addition of $\mathbb{U}$ to intuitonistic logic, IL. We will study this combination using Kripke frames. Before defining the frames, it is worth noting that there have been studies of S5-ish logics over IL. Ono (1977) considers different S5-type axioms in the context of IL, which gives rise to a range of different logics. It follows from results in section 5.3 of Niki & Omori (2020) that the present logic $\mathsf{IL}^{\mathbb{U}}$ is equivalent to Ono's L4—the strongest S5-type logic considered in Ono (1977).

Definition 1 (Kripke frame for IL)**.** *A Kripke frame for* IL *is a pair* $\langle W, \leq \rangle$ *such that*

- *W is non-empty, and*
- *$\leq$ is a reflexive, transtive binary relation on W.*

[1] As an example of the interesting features, S5 has a connection to monadic quantified logics. This connection was noted by Halmos (2016) in the context of monadic algebra, and Mints (1992) proves the coicidence for the case of classical logic. Bull (1966) proves it for the case of intuitionistic logic. Caicedo et al. (2019) show it for Dummett's logic LC. Ferenz (2024) shows this for certain S5-ish extensions of some relevant logics.

[2] These logics are in the framework FMLA of Humberstone (2011).

A Kripke model M *is a triple* $\langle W, \leq V\rangle$ *where* $\langle W, \leq\rangle$ *is a frame and* V *is a function from* At *to sets of worlds such that if* $x \in V(p)$ *and* $x \leq y$, *then* $y \in V(p)$.

The verification relation $\Vdash$ is defined over the whole language as follows.

- $w \Vdash p$ iff $w \in V(p)$.
- $w \Vdash B \wedge C$ iff $w \Vdash B$ and $w \Vdash C$.
- $w \Vdash B \vee C$ iff $w \Vdash B$ or $w \Vdash C$.
- $w \Vdash B \rightarrow C$ iff for all $x \in W$ such that $w \leq x$ and $x \Vdash B$, $x \Vdash C$.
- $w \Vdash \neg B$ iff for all $x \in W$ such that $w \leq x$, $x \nVdash B$.
- $w \Vdash \mathbb{U}B$ iff for all $x \in W$, $x \Vdash B$.

A consequence of the definitions is that the heredity condition extends to arbitrary formulas. We will state this, without proof.

Lemma 2 (Heredity). *For all Kripke models* M, *all* $x, y \in W$, *and all formulas* A, *if* $x \leq y$ *and* $x \Vdash A$, *then* $y \Vdash A$.

There is a special class of formulas that we will examine throughout this paper.

Definition 3 ($\mathbb{U}$-formula). *A formula* A *is a* $\mathbb{U}$*-formula iff is of the form* $\mathbb{U}B$.

The $\mathbb{U}$-formulas are a special case of a more general class of formulas, namely the modalized formulas. A formula A is *modalized* iff every atom in A occurs in the scope of $\mathbb{U}$. Modalized formulas are, in many ways, distinctive, and they are generally important in the study of modal logics.[3] Nonetheless, we only need to consider the special case of $\mathbb{U}$-formulas for the results of interest to this paper.

The main feature of universal necessity on which we will rely is one that is common across the different models we will look at.

Lemma 4 (Ubiquity). *In a model* M, *if* $w \Vdash \mathbb{U}A$, *for some* $w \in W$, *then* $u \Vdash \mathbb{U}A$, *for all* $u \in W$.

Proof. Suppose $w \Vdash \mathbb{U}A$. Then it follows that for all $u \in W$, $u \Vdash A$, so $x \Vdash \mathbb{U}A$, for arbitrary x. □

In the terminology of Standefer (2022; 2023), formulas with $\mathbb{U}$ as their main connective are *ubiquitous*, either true at all points in a model or true at no points in a model. $\mathbb{U}A$ either holds everywhere or nowhere. This feature gives rise to a surprising tension with a distinctive feature of intuitionistic logic, the disjunction property.

[3] Mints (1992) uses modalized formulas in his discussion of the connection between S5 and monadic quantification. Prawitz (1965) uses modalized formulas in his natural deduction systems for S5.

Definition 5 (Disjunction property). *A logic* L *has the disjunction property iff if* $A \vee B$ *is valid, then* A *is valid or* B *is valid.*

IL enjoys the disjunction property. The disjunction property is an important part of what is meant by saying that IL is a *constructive* logic. When considering extensions of intuitionistic logic with new connectives, the extension enjoying the disjunction property is one indication that the new connective aligns with intuitionistic logic. Adding $\mathbb{U}$ to IL results in violations of the disjunction property. It is easiest to see this using some lemmas.

Lemma 6. *For all Kripke models* M*, for all* $w \in W$*,* $w \Vdash \neg\mathbb{U}A$ *iff* $w \nVdash \mathbb{U}A$*.*

Proof. The left to right direction is immediate.

Next, suppose that $w \nVdash \mathbb{U}A$. By lemma 4, this implies that for all $x \in W$, $x \nVdash \mathbb{U}A$. Therefore, for all $x \in W$ such that $w \leq x$, $x \nVdash \mathbb{U}A$, whence $w \Vdash \neg\mathbb{U}A$, as desired. □

For the special case of $\mathbb{U}$-formulas, intuitionistic negation collapses to that of classical logic.

Corollary 7. *In the class of Kripke frames for* IL*,* $\mathbb{U}A \vee \neg\mathbb{U}A$ *is valid.*

Proof. Let M be a model on a Kripke frame and let $w \in W$. Since $w \Vdash \mathbb{U}A$ or $w \nVdash \mathbb{U}A$ from lemma 6 it follows that $w \Vdash \mathbb{U}A$ or $w \Vdash \neg\mathbb{U}A$. Therefore, $w \Vdash \mathbb{U}A \vee \neg\mathbb{U}A$. □

Given that it is easy to see that neither $\mathbb{U}p$ nor $\mathbb{U}\neg p$, for example, are valid, it follows that the addition of $\mathbb{U}$ leads to widespread failure of the disjunction property.

We can obtain another corollary of lemma 6, demonstrating a collapse of the intuitionistic implication to the classical material conditional for $\mathbb{U}$-formulas.

Corollary 8. *For all Kripke models* M *for* IL*, for all* $w \in W$*,* $w \Vdash (\mathbb{U}A \rightarrow \mathbb{U}B) \leftrightarrow (\neg\mathbb{U}A \vee \mathbb{U}B)$*.*

Proof. The left to right direction holds for all formulas in IL already, so we will prove the converse.

Suppose that for some $x \in W$, $w \leq x$, $x \Vdash \mathbb{U}A \rightarrow \mathbb{U}B$, and $x \nVdash \neg\mathbb{U}A \vee \mathbb{U}B$. Therefore, $x \nVdash \neg\mathbb{U}A$ and $x \nVdash \mathbb{U}B$. By the preceding corollary, $x \nVdash \neg\mathbb{U}A$ implies $x \Vdash \mathbb{U}A$, so it follows that $x \Vdash \mathbb{U}B$. This is a contradiction, so $x \Vdash \neg\mathbb{U}A \vee \mathbb{U}B$, which establishes the desired claim. □

This corollary demonstrates that, for a certain class of formulas, the intuitionistic implication collapses the classical material conditional, namely the conditional $\mathbb{U}A \rightarrow \mathbb{U}B$ defined as $\neg\mathbb{U}A \vee \mathbb{U}B$. While this holds for certain choices of A and B in IL, namely when A and B are theorems, the are no restrictions on A and B in the corollary above.

To sum up this section, we have, in $\mathsf{IL}^{\mathbb{U}}$ violations of key features of IL that are important to proponents of IL. This is a story we will see repeated below.

3 First-Degree Entailment

The logic FDE, or first-degree entailment, is, perhaps, the most well known four-valued logic.[4] It was isolated as first-degree fragment of the relevant logic E, meaning the set of formulas $A \to B$ where A and B do not contain $\to$, by Anderson and Belnap. It has since been found to be the first-degree fragment of all the standard relevant logics.

FDE has been a focus of study both on its own and in the context of relevant logics. Omori and Wansing (2017) provide a good overview of work on FDE. Beall (2017; 2018) has argued that FDE is the basic subclassical logic. Levesque (1984) uses FDE in a logic of awareness, and Standefer et al. (2023) argue that FDE should be used for logical closure of justifications.

We will focus on a few specific features of FDE to be presented shortly. We will study the star frames for FDE similar to those of Routley and Routley (1972).[5]

Definition 9 (Star frame for FDE). *A star frame for* FDE *is a pair* $\langle K, ^* \rangle$ *where*

- K *is non-empty, and*
- $a^{**} = a$.

A star model M *is a triple* $\langle K, ^*, V \rangle$ *such that* $\langle K, ^* \rangle$ *is a star frame and* V *is a function from* At *to sets of worlds.*

Such a star model is said to be built on the star frame.

In a given model, the verification relation $\Vdash$ is defined over the language without $\to$ as follows.

- $a \Vdash p$ iff $w \in V(p)$.
- $a \Vdash B \land C$ iff $a \Vdash B$ and $a \Vdash C$.
- $a \Vdash B \lor C$ iff $a \Vdash B$ or $a \Vdash C$.
- $a \Vdash \neg B$ iff $a^* \nVdash B$.
- $a \Vdash \mathbb{U}B$ iff for all $b \in K$, $b \Vdash B$.

We are evaluating formulas at worlds only if they lack the implication connective.

[4] See Dunn (1966) or Belnap (1977a; 1977b). Anderson and Belnap (1975) cover FDE in chapter 3. The collection Omori and Wansing (2019) contains this material, as well as many other papers on FDE.

[5] See also Dunn (1966).

Definition 10 (Holding, validity). *For all formulas* A, B*,* $A \to B$ *is holds in a star model iff for all* $a \in K$*, if* $a \Vdash A$*, then* $a \Vdash B$*.*

For all formulas A, B*,* $A \to B$ *is valid in the class of star frames iff* $A \to B$ *holds in any model built on a star frame.*

Validity is defined in a restricted form, namely only for formulas of the form $A \to B$ where A and B do not contain $\to$, because we are interested in the valid first-degree formulas.

There are some features of FDE that are well-known and generally regarded as virtues of FDE, which we will state here without proof.

- FDE is paraconsistent, i.e. there is no A such that for all B is $(A \wedge \neg A) \to B$ valid.
- FDE is paracomplete, i.e. there is no A such that for all B is is $B \to (A \vee \neg A)$ valid.
- FDE enjoys variable-sharing, i.e. if $A \to B$ is valid, then A and B share an atom.

All three of these features are violated in $\mathsf{FDE}^{\mathbb{U}}$. To show this, we note that lemma 4 carries over to star models.

Lemma 11. *In a star model* M*, if* $a \Vdash \mathbb{U}A$*, for some* $a \in K$*, then* $b \Vdash \mathbb{U}A$*, for all* $b \in K$*.*

Proof. The proof is essentially the same as that of lemma 4. □

This lemma has consequences for negation in star models.[6]

Lemma 12. *In a star model* M*,* $a \Vdash \neg\mathbb{U}A$ *iff* $a \nVdash \mathbb{U}A$*.*

Proof. Let M be a star model. Suppose that $a \Vdash \neg\mathbb{U}A$. It follows that $a^* \nVdash \mathbb{U}A$. From lemma 11, this implies $a \nVdash \mathbb{U}A$.

For the converse, suppose $a \nVdash \mathbb{U}A$. By lemma 11, we have $a^* \nVdash \mathbb{U}A$, so $a \Vdash \neg\mathbb{U}A$. □

We can then obtain the following two corollaries.

Corollary 13. *Let* M *be a star model. Then for all* $a \in K$*,* $a \Vdash \mathbb{U}A \vee \neg\mathbb{U}A$*.*

Corollary 14. *Let* M *be a star model. Then for all* $a \in K$*,* $a \nVdash \mathbb{U}A \wedge \neg\mathbb{U}A$*.*

These suffice for the following theorem.

[6]A further similarity with the intuitionistic case is illuminated if we work with models closer to those of Routley and Routley (1972) which have a designated state $g \in K$ where $g = g^*$. In this setting we can, following Routley and Routley (1972, p.348), define an analogue of classical S5 strict-implication by saying that $A \dashv B$ holds in a star model iff for all $a \in K$ where $a = a^*$, if $a \Vdash A$, then $a \Vdash B$. In this setting Lemma 4 shows that in we have a collapse between the entailments and the strict-entailments between $\mathbb{U}$-formulas, similar to the way in which in $\mathsf{IL}^{\mathbb{U}}$ we have a collapse between the intuitionistic and classical implication over $\mathbb{U}$-formulas (as recorded in Corollary 8).

Theorem 15. *For all formulas* A *and* B*, both*

- $A \rightarrow (\mathbb{U}B \vee \neg\mathbb{U}B)$ *and*
- $(\mathbb{U}B \wedge \neg\mathbb{U}B) \rightarrow A$

are valid in the class of all star frames.

Proof. For the first, note that by corollary 13, the consequent holds at all points in all models. There can, then, be no counterexamples. Similarly, for the second, note that by corollary 14, the antecedent fails at all points in all models, so there can, similarly, be no counterexamples. □

Corollary 16. $\mathsf{FDE}^{\mathbb{U}}$ *does not enjoy variable-sharing.*

Proof. By the preceding theorem, $p \rightarrow (\mathbb{U}q \vee \neg\mathbb{U}q)$ is valid. □

Thus, see that three of the key features of FDE fail in the context of $\mathsf{FDE}^{\mathbb{U}}$. Much as in $\mathsf{IL}^{\mathbb{U}}$, violations of the highlighted features arise in the context of $\mathbb{U}$-formulas. In the next section, we will turn to relevant logics to further extend our results.

4 Relevant Logics

In this section, we turn to relevant logics. Relevant logics are substructural logics whose implication connectives are supposed to enforce a strong connection between antecedent and consequent.[7] A key feature of relevant logics is that they enjoy the *variable-sharing* property, which says that if $A \rightarrow B$ is valid, then A and B share an atom.[8] There are many relevant logics, but we will focus on the weak relevant logic B. For the most part, the particular choice of which relevant logic to use will not make a difference, for reasons that will become clearer below.

We will study relevant logics using Routley-Meyer ternary relational frames.[9]

Definition 17 (Routley-Meyer frame)**.** *A Routley-Meyer frame is a quadruple* $\langle K, N, R, ^* \rangle$*, where* $K \neq \emptyset$*,* $N \subseteq K$*,* $R \subseteq K^3$*, and* $^* : K \mapsto K$*, that obeys the following conditions, where* $a \leq b =_{Df} \exists x \in N\, Rxab$*,*

[7] See Dunn and Restall (2002), Bimbó (2007), Mares (2020), Logan (2024), or Standefer (202x) for an overview of relevant logics.

[8] See Standefer (2024) for discussion.

[9] Routley-Meyer models were first introduced by Routley and Meyer (1972a; 1973; 1972b). See Bimbó et al. (2018) for discussion of the early development of Routley-Meyer models.

(B1) $\leq$ *is a partial order,*

(B2) $a^{**} = a$*,*

(B3) $a \leq b$ *only if* $b^* \leq a^*$*,*

(B4) *if* $d \leq a$*,* $e \leq b$*,* $c \leq f$*, and* $Rabc$*, then* $Rdef$*, and*

Definition 18 (Routley-Meyer model)**.** *A Routley-Meyer model* M *is a quintuple* $\langle K, N, R, ^*, V\rangle$ *where the first four components make up a frame and* V *is a function from* At *to subsets of* K *obeying the condition that if* $a \in V(p)$ *and* $a \leq b$*, then* $b \in V(p)$*.*

The model $M = \langle K, N, R, ^*, V\rangle$ *is said to built on the frame* $F = \langle K, N, R, ^*\rangle$*.*

The verification relation $\Vdash$ is defined over the whole language as follows.

- $a \Vdash p$ iff $a \in V(p)$
- $a \Vdash B \wedge C$ iff $a \Vdash B$ and $a \Vdash C$
- $a \Vdash B \vee C$ iff $a \Vdash B$ or $a \Vdash C$
- $a \Vdash B \rightarrow C$ iff for all $b, c \in K$, if $Rabc$ and $b \Vdash B$, then $c \Vdash C$
- $a \Vdash \neg B$ iff $a^* \nVdash B$
- $a \Vdash \mathbb{U}B$ iff for all $b \in K$, $b \Vdash B$

Given these definitions, we can define holding in a model and validity in a class of frames.

Definition 19. *A formula* A *holds in a model* M *iff for all* $a \in N$, $a \Vdash A$*.*

A formula A *is valid in a class of models iff for all frames* F *in the class,* A *holds in* M*, for all* M *built on* F*.*

The class of all basic Routley-Meyer frames gives us the logic B.

Our interest will be in the logic $B^{\mathbb{U}}$. As with $IL^{\mathbb{U}}$, $\mathbb{U}$-formulas have several important properties.

Lemma 20. *In a Routley-Meyer model* M*, if* $a \Vdash \mathbb{U}A$*, for some* $a \in K$*, then* $b \Vdash \mathbb{U}A$*, for all* $b \in K$*.*

Proof. The proof is essentially the same as in the IL case. $\square$

While heredity is postulated for atoms, it extends to the full language, as in $IL^{\mathbb{U}}$.

Lemma 21 (Heredity)**.** *For all Routley-Meyer models* M*, all* $a, b \in K$*, and all formulas* A*, if* $a \leq b$ *and* $a \Vdash A$*, then* $b \Vdash A$*.*

Proof. The proof is by induction on the structure of A. The only new case is when A is of the form $\mathbb{U}B$. The case is covered by lemma 20. □

Using this lemma, we can prove another lemma, whose proof we will omit as it is standard.

Lemma 22 (Verification)**.** *For all models* M*, the following are equivalent.*

- *For all* $a \in K$*, if* $a \Vdash A$*, then* $a \Vdash B$*.*
- $A \to B$ *holds in* M*.*

The verification lemma drastically simplifies the verification of formulas $A \to B$ in a model, especially when A and B do not contain any implications. That suggests a connection to star frames, which bears out. Given a Routley-Meyer frame $\langle K, N, R, ^* \rangle$ its reduct $\langle K, ^* \rangle$ is a star frame. Therefore, we can obtain a corollary of lemma 20 concerning negation, as in $\mathsf{FDE}^{\mathbb{U}}$.[10]

Corollary 23. *In a Routley-Meyer model* M*,* $a \Vdash \neg\mathbb{U}A$ *iff* $a \nVdash \mathbb{U}A$*.*

As a consequence of the preceding corollary, we obtain the validity of some formulas that can violate variable-sharing.

Corollary 24. *The formulas* $(\mathbb{U}A \wedge \neg\mathbb{U}A) \to B$ *and* $B \to (\mathbb{U}A \vee \neg\mathbb{U}A)$ *are valid in the class of all Routley-Meyer frames.*

Proof. The proof is essentially the same as the proof of the similar fact in the previous section. □

Corollary 25. $\mathsf{B}^{\mathbb{U}}$ *violates variable-sharing.*

There are other violations of variable-sharing in $\mathsf{B}^{\mathbb{U}}$ that are not in $\mathsf{FDE}^{\mathbb{U}}$. Two examples are $\mathbb{U}p \to (\mathbb{U}q \to \mathbb{U}p)$ and $\mathbb{U}q \to (\mathbb{U}p \to \mathbb{U}p)$, which are instances of the paradoxes of implication, paradoxes relevant logics were meant to avoid. As with $\mathsf{IL}^{\mathbb{U}}$, we get a collapse of the implication to the material conditional.

Lemma 26. *For all formulas* A *and* B*,* $(\neg\mathbb{U}A \vee \mathbb{U}B) \to (\mathbb{U}A \to \mathbb{U}B)$ *is valid.*

[10] We are adopting an Australian-plan approach to negation here. The Australian plan has been defended recently by Berto (2015) and Berto and Restall (2019). The rival American plan for negation has recently been defended by De and Omori (2018) and Omori and De (2022). While in many contexts, the two plans give equivalent results, in the context of universal necessity, there are some differences that can emerge with respect to valid formulas. We would like to thank Dave Ripley and Hitoshi Omori for discussion of this point.

Proof. Let M be a Routley-Meyer model, and suppose $a \Vdash \neg\mathbb{U}A \vee \mathbb{U}B$. Then, either $a \Vdash \neg\mathbb{U}A$ or $a \Vdash \mathbb{U}B$. Suppose $a \Vdash \neg\mathbb{U}A$. Further, suppose that $Rabc$ and $b \Vdash \mathbb{U}A$. From $a \Vdash \neg\mathbb{U}A$ and lemma 20, it follows that $b \nVdash \mathbb{U}A$, which is a contradiction, so, classically, $c \Vdash \mathbb{U}B$. Next, suppose that $a \Vdash \mathbb{U}B$. Suppose further that $Rabc$ and $b \Vdash \mathbb{U}A$. From lemma 20 and the supposition of $a \Vdash \mathbb{U}B$, it follows that $c \Vdash \mathbb{U}B$. In both cases, we have established $c \Vdash \mathbb{U}B$, which suffices for $a \Vdash \mathbb{U}A \rightarrow \mathbb{U}B$. □

While the implication from material conditional to non-classical implication holds generally in **IL**, it does not hold generally in any of the standard relevant logics. The converse implication, from relevant implication to material conditional, holds for $\mathbb{U}$-formulas, under an additional supposition.

Lemma 27. *Let $\mathcal{C}$ be the class of Routley-Meyer frames such that for all $a \in K$, there are $b, c \in K$ such that $Rabc$. Then for all formulas A, B, $(\mathbb{U}A \rightarrow \mathbb{U}B) \rightarrow (\neg\mathbb{U}A \vee \mathbb{U}B)$ is valid in $\mathcal{C}$.*

Proof. Let M be a model built on a Routley-Meyer frame in $\mathcal{C}$. Suppose that $a \Vdash \mathbb{U}A \rightarrow \mathbb{U}B$. Suppose, for reductio, that $a \nVdash \neg\mathbb{U}A \vee \mathbb{U}B$. This implies that that $a \nVdash \neg\mathbb{U}A$. It follows that $a \Vdash \mathbb{U}A$. From the assumption, there $b, c \in K$ such that $Rabc$. By lemma 20 and $a \Vdash \mathbb{U}A$, we can conclude $b \Vdash \mathbb{U}A$. This together with the supposition $a \Vdash \mathbb{U}A \rightarrow \mathbb{U}B$ entails that $c \Vdash \mathbb{U}B$. By lemma 20 again, $a \Vdash \mathbb{U}B$, whence $a \Vdash \neg\mathbb{U}A \vee \mathbb{U}B$, which is a contradiction. □

As far as we can tell, the restriction of the lemma is essential for the proof to work. The condition, that for all $a \in K$, there are $b, c \in K$ such that $Rabc$, is known as R-*seriality*.[11] While R-seriality is not built in the Routley-Meyer frames for B, it can be assumed without changing the logic. The class of Routley-Meyer frames obeying R-seriality also generates B. The reason is that, in Henkin-style canonical model completeness proofs for the standard relevant logics, the canonical frame is R-serial. The condition does make a difference once $\mathbb{U}$ is in the language, so it is not uncontroversial. The preceding result shows that, for the R-serial frames, for $\mathbb{U}$-formulas, the relevant implication collapses to the classical material conditional, which should be a deeply unappealing result to the relevant logician. The frames for some relevant logics will obey R-seriality, such as those that contain $(A \wedge (A \rightarrow B)) \rightarrow B$, whose frame condition is $\forall x \in K,\ Rxxx$. The possibility of avoiding R-seriality only arises for some of the weaker, contraction-free relevant logics.

[11] R-seriality comes up in the discussion of Halldén completeness for relevant modal logics. See Mares (2003) and Seki (2015) for more on Halldlén completeness for relevant logics.

5 Classicality

We have introduced three different logics, defined using three different kinds of frames. These logics all have distinctive non-classical features that are taken to be virtues by their proponents. We have seen that the result of adding the universal necessity operator '$\mathbb{U}$' to the language results in violations of these features. In the case of IL and B, the addition of $\mathbb{U}$ results in a collapse of the non-classical implication to the classical material conditional, at least for $\mathbb{U}$-formulas.

Many authors in the relevant logic litearture have viewed universal necessity as a deeply classical concept. Parks and Byrd (1989, p.180) remark that among relevant logicians there was a "feeling that the Leibnizian view of necessity is incompatible with key tenets of relevance logic." By "Leibnizian view of necessity," they mean universal necessity. Standefer (2023) takes the fact that adding $\mathbb{U}$ to relevant logics results in violations of variable-sharing to vindicate the suspicion voiced by Parks and Byrd. We want to go further. Universal necessity is incompatible with key tenets of relevant logic because it smuggles in a kind of covert classicality. To put it loosely, $\mathbb{U}$-formulas behave similarly to formulas of classical logic.

To illustrate what we mean, consider B. Unlike some relevant logics, no classical tautologies in the vocabulary $\{\neg, \vee, \wedge\}$ are valid for B. As we saw in the previous section, $\mathbb{U}A \vee \neg\mathbb{U}A$ is valid for $B^{\mathbb{U}}$. Due to De Morgan equivalences and the double negation laws that are built into B, it follows that $\neg(\mathbb{U}A \wedge \neg\mathbb{U}A)$ is also valid. Next, note that $B^{\mathbb{U}}$ is closed under the rule of disjunctive syllogism for $\mathbb{U}$-formulas,

- if $\mathbb{U}A$ is valid and $\neg\mathbb{U}A \vee \mathbb{U}B$ is valid, then $\mathbb{U}B$ is valid.

It follows that all substitution instances of classical tautologies in the vocabulary $\{\neg, \wedge, \vee\}$ where the atoms are replaced by $\mathbb{U}$-formulas will be valid. A form of classical logic emerges, even in the austere setting of Routley-Meyer frames for B. But, one of the reasons non-classical logicians want to use non-classical logics is because there is something about classical logic that we want to avoid. The classical tautologies reemerging against the backdrop of our favored non-classical logics is not a welcome result.

The problem stems, we think, from the way we model non-classical logics and how that interacts with universal necessity. In frame-based models for non-classical logics, we often obtain non-classical, or generally intensional, logical behavior from a connective by attending to how a subformula is evaluated at potentially distinct points. The approach to negation in star models will serve as an example. A formula, A, and its negation, $\neg A$, can both hold at a point because A fails at the star of the point. When considering the basic vocabulary of non-classical logics, we are restricted to a local view of the model, able to consider only certain points, while there may be other points in the model that are not considered.[12] Again, consider negation in star models. In evaluating $\neg p$ at a point b, we consider only whether p holds at b^*. For that evaluation, assuming $b^* \neq b$, whether p holds at b is immaterial.

When universal necessity enters the picture, one can take a global view on the model. When that happens, the local view is lost. $\mathbb{U}$-formulas hold either at all points or at none. Let us, for a moment, take sets of points as propositions, as is common for models for non-classical logics, and take the set of points where a formula holds to be its proposition. We can see that the only propositions available for $\mathbb{U}$-formulas are the set of all points and the empty set. This limited set of options is very close to the usual two truth-values of classical logic.

We think that this idea is on the right track, and in future work we hope to make this idea both precise and general to prove that against the backdrop of models for many non-classical logics, the addition of $\mathbb{U}$ will result in classical logic reemerging. For now, we will have to make do with the suggestive examples. A consequence of this classicality idea is that we expect that the addition of $\mathbb{U}$ to other non-classical logics will result in violations of the salient non-classical features of the underlying logic.[13]

To close, we will note a curious feature of $\mathbb{U}$. Despite the arguably negative consequences that we have seen of adding $\mathbb{U}$ to a logic defined in terms of frames, there is a silver lining. One can interpret $\mathbb{U}$ in any model on any frame of the classes we considered, since one can do so without adding an accessibility relation to the frame. Alternatively, one can trivially add a universal binary relation. This means that the addition of $\mathbb{U}$ is *conservative* over the base logic, so that if a formula A that does not contain $\mathbb{U}$ is valid in the class of the frames with $\mathbb{U}$ in the language, then A is valid in the class of frames for the base logic. The addition of $\mathbb{U}$, then, does not disturb the base logic, although it does introduce a form of classical logic, which in the eyes of many of the early relevant logicians would seemingly make it a conservative extension in a different sense also.

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[12] The idea of local and global views of a model is common in classically-based modal logic. See, for example, comments about locality by Blackburn et al. (2002, 67ff., 417)

[13] We thank Hitoshi Omori and Andrzej Indrzejczak for discussion of this point.

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